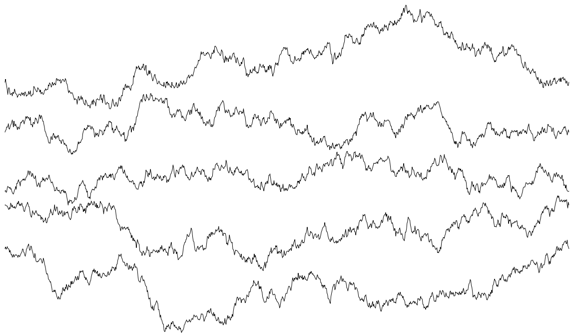
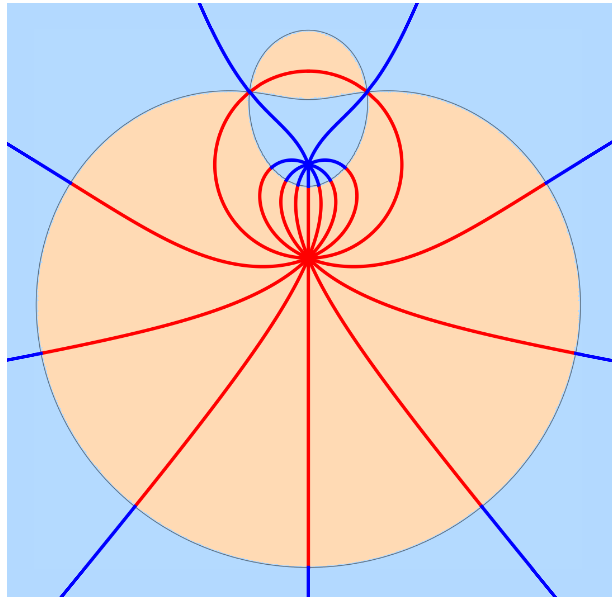
KPZ fluctuations in finite volume

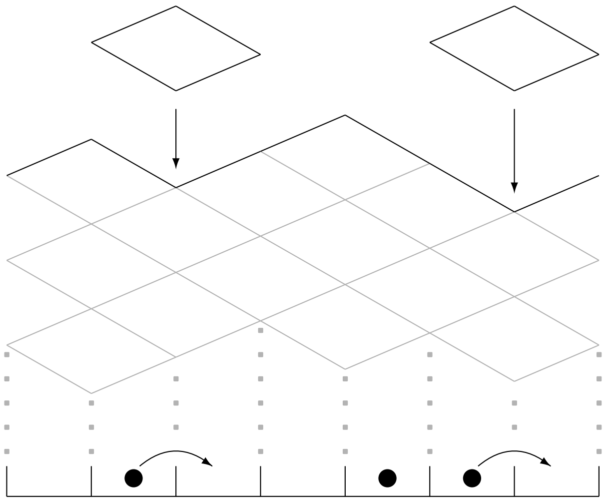


Sylvain Prolhac

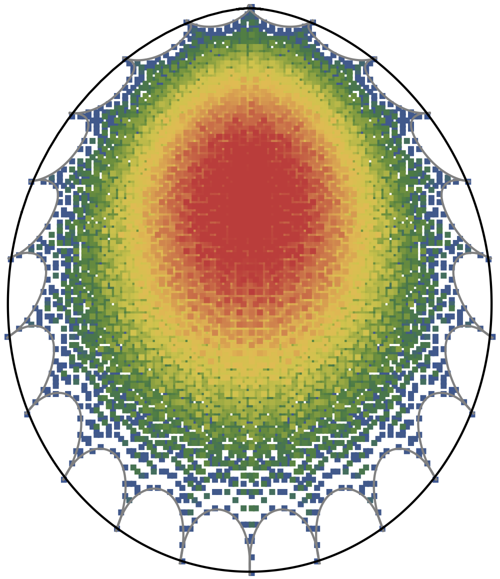
Soutenance HDR

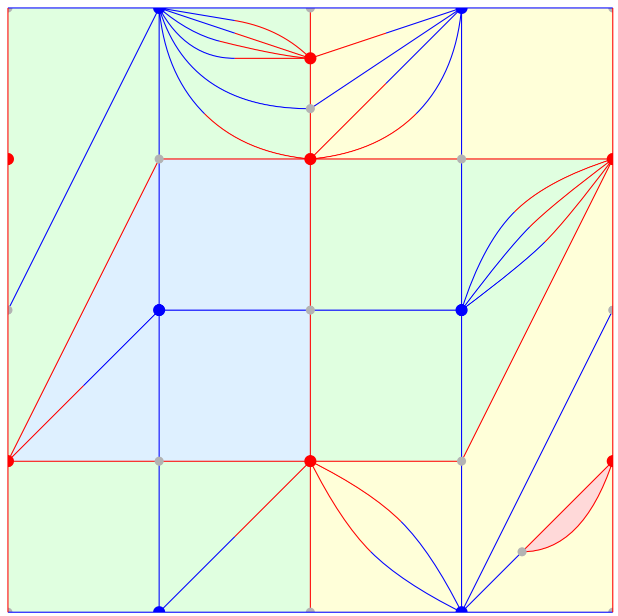
15 janvier 2024





$t \sim L^{3/2}$





Outline of my research

Statistical physics of systems with large number of degrees of freedom

Equilibrium   micro-states $\mathcal{C}$ weighted by $\mathrm{e}^{-E(\mathcal{C})/k_{\mathrm{B}}T}$

Non-equilibrium phenomena   large scale currents, irreversibility
dynamics $\Rightarrow$ time-dependent statistics

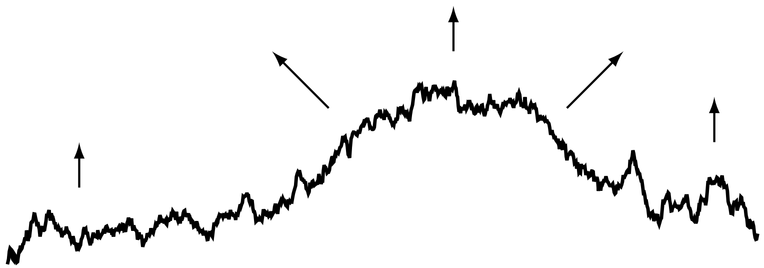
Prominent example: KPZ universality in 1+1 dimension
Microscopic details do not matter $\Rightarrow$ exactly solvable discrete models

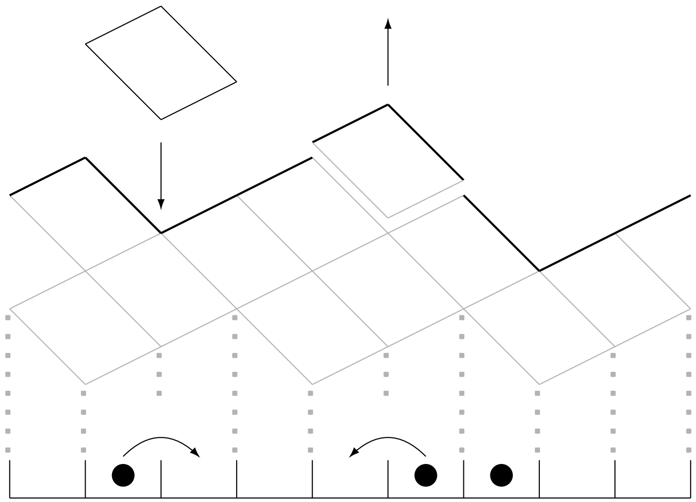
Before 2012: stationary large deviations (late time regime ; PhD)
dynamics of an infinite system (early time regime ; postdoc)

Since 2012 (MCF at LPT): relaxation of fluctuations in finite volume
initial condition $\rightarrow$ non-equilibrium stationary state

Past few years: developed a Riemann surface approach for this problem

I Various settings for KPZ fluctuations





II KPZ universality, finite volume effects

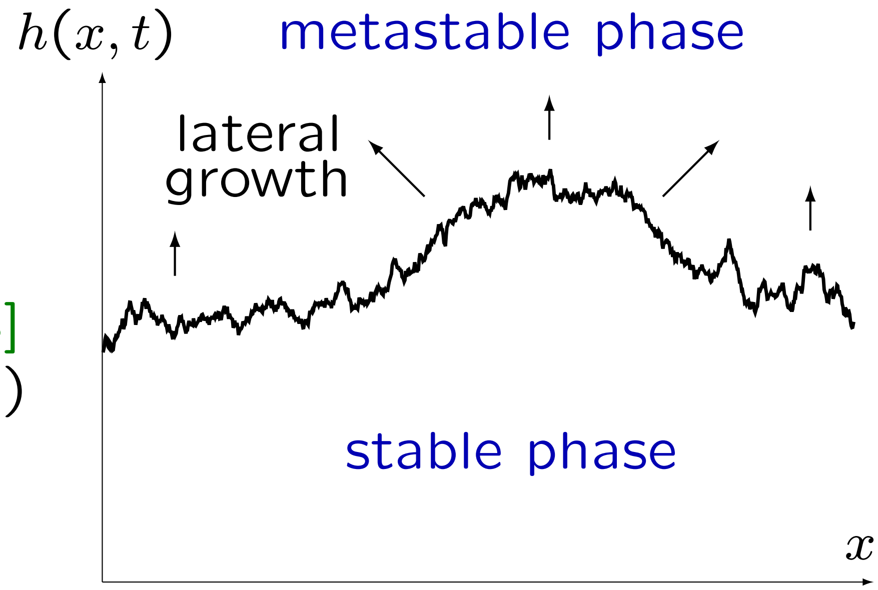
III Riemann surface approach

Interface growth

KPZ equation [Kardar-Parisi-Zhang 1986]

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi$$

smoothing growth noise



Rigorous mathematical definition [Hairer 2013]
Discrete models (Eden, ballistic deposition, ...)

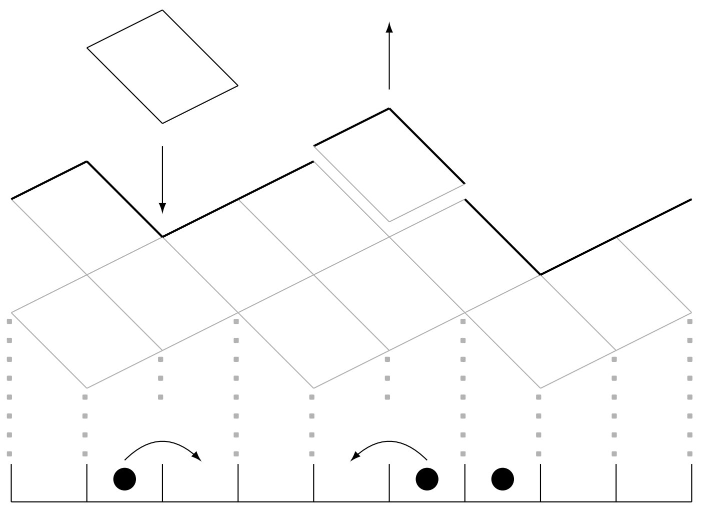
Dynamics at large scales $\rightarrow$ KPZ fixed point
Universal scale invariant object
$\hookrightarrow$ exact exponents, correlation functions without fitting parameters

Other universality classes if too much smoothing
strong quenched disorder $\xi(x,t) \rightarrow \xi(x,h)$
long range correlated noise

Experimental observations (mainly early growth: $x \in \mathbb{R}$, no boundary effects)

- colonies of bacteria [Matsushita et al. 1998] and cells [Mazarei et al. 2022]
- slow combustion of paper [Maunuksela et al. 1997, Miettinen et al. 2005]
- turbulent phases of a liquid crystal [Takeuchi et al. 2010, 2020]
- reaction fronts driven through porous medium [Atis et al. 2015]

Driven particles in 1 dimension



Discrete models

slope $\sigma = \partial_x h$ $\leftrightsquigarrow$ density fluctuations

KPZ fluctuations for time-integrated current

Conservation law

KPZ equation	slope $\sigma = \partial_x h$	Burgers' equation
$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi$	$\Rightarrow$	$\partial_t \sigma = \partial_x^2 \sigma + 2\sigma\,\partial_x \sigma + \partial_x \xi$

superdiffusive anomalous transport

current $J[\sigma] = \partial_x \sigma + \sigma^2 + \xi$

Fluids with few conservation laws (e.g. mass, energy, momentum)

$\Rightarrow$ coupled Burgers' equations

Non-linear fluctuating hydrodynamics [Spohn 2014] $\Rightarrow$ expansion into normal modes at late times

KPZ sound modes propagating (+ diffusive heat modes, higher universality classes)

Quantum systems

Quantum fluids

non-linear fluctuating hydrodynamics for local fluctuations of conserved fields

- Gross-Pitaevskii equation [Kulkarni et al. 2015]
- Heisenberg spin chain [Ljubotina et al. 2019]

Quantum dynamics subjected to classical noise

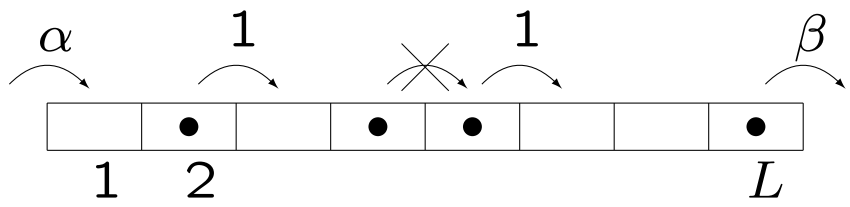
$\hookrightarrow$ KPZ fluctuations for the entanglement entropy

- random unitary dynamics [Nahum et al. 2017]
- continuous monitoring of a quantum system [Weinstein et al. 2022]

Experimental observations of KPZ fluctuations

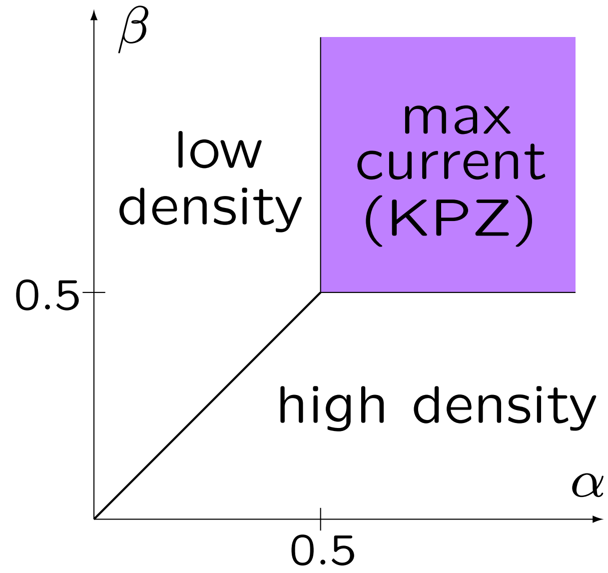
- low energy spectrum antiferromagnet probed with neutron scattering [Scheie et al. 2021]
- superdiffusive spin transport for cold atoms trapped in optical lattice [Wei et al. 2022]
- coherence decay 1d driven polariton condensate in semiconductor microcavity [Fontaine et al. 2022]

Exactly solvable models



(T)ASEP: (totally) asymmetric simple exclusion process

Boundaries: periodic, finite density of particles
open, fixed slope $\partial_x h$



KPZ time scale $t \sim L^{3/2}$

generator ~ Hamiltonian XXZ spin chain

$H_{\mathrm{XXZ}} = \sum_i (S_i^{\mathrm{x}} S_{i+1}^{\mathrm{x}} + S_i^{\mathrm{y}} S_{i+1}^{\mathrm{y}} + \Delta\, S_i^{\mathrm{z}} S_{i+1}^{\mathrm{z}})$

Replica solution KPZ equation

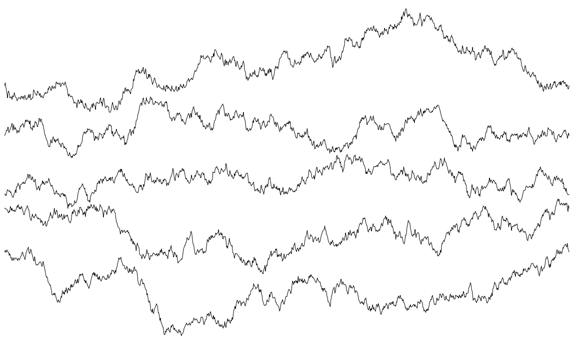
mapping to Lieb-Liniger δ-Bose gas (with attractive interaction)

$$\langle Z(x,t)^n \rangle = \langle x, \dots, x | \mathrm{e}^{-tH_n} | \psi_0 \rangle \quad \text{with} \quad H_n = -\frac{1}{2} \sum_{j=1}^{n} \partial_x^2 - \sum_{i<j} \delta(x_i - x_j)$$

Hierarchy of various models: directed polymers, interacting Brownian motions, vertex models, random tilings, polynuclear growth, ...

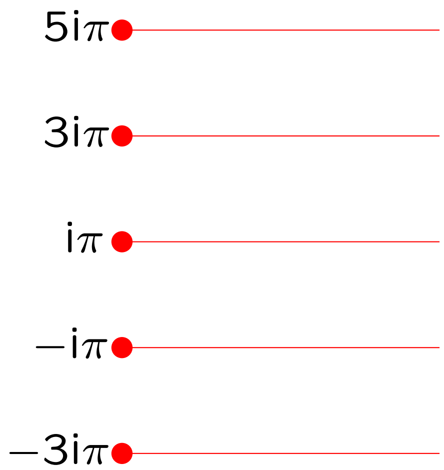
I Various settings for KPZ fluctuations

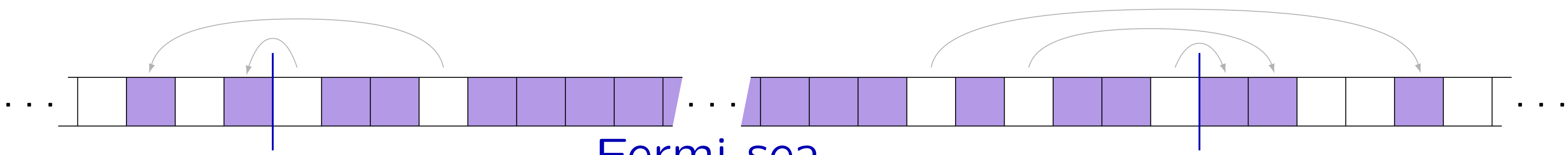
II KPZ universality, finite volume effects



$\mathrm{Li}_{5/2}$

analytic continuation





Fermi sea

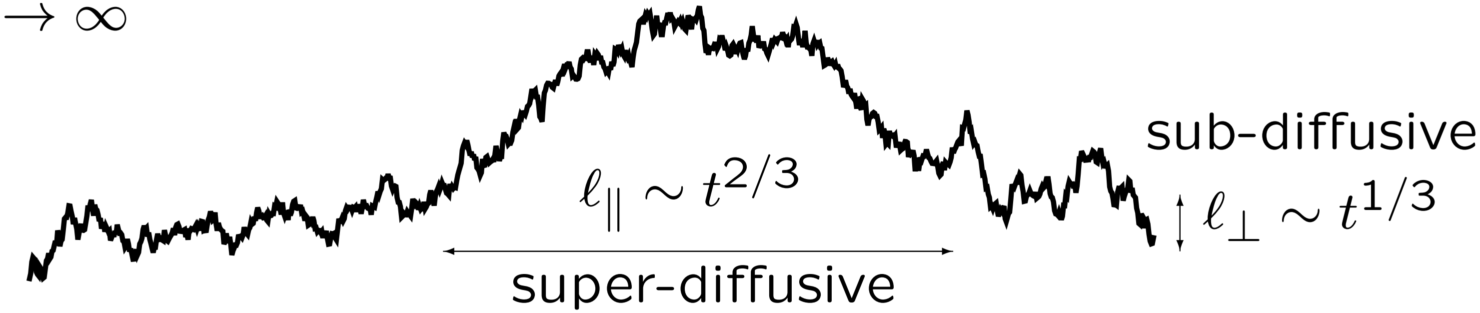
III Riemann surface approach

Exponents and scaling functions at the KPZ fixed point

Infinitely large system, $t \to \infty$

Dynamical length scales

$$\ell_{\parallel} \sim t^{1/z} = t^{2/3}$$
$$\ell_{\perp} \sim t^{\alpha/z} = t^{1/3}$$



Power laws characterized by universal exponents
$z = 3/2$ dynamical exponent
$\alpha = 1/2$ roughness exponent

Also universal scaling functions: probability distributions, correlation functions (after subtracting non-universal global velocity, and rescaling)

examples :

- flat initial condition $\Rightarrow \mathbb{P}\left(\frac{h(x,t)}{t^{1/3}} \le s\right) \underset{t\to\infty}{\to} F_{\mathrm{GOE}}(s)$
- curved initial condition $\Rightarrow \mathbb{P}\left(\frac{h(x,t)}{t^{1/3}} \le s\right) \underset{t\to\infty}{\to} F_{\mathrm{GUE}}(s)$

F_{GOE} / F_{GUE} distribution extremal eigenvalue random matrices

Finite volume effects

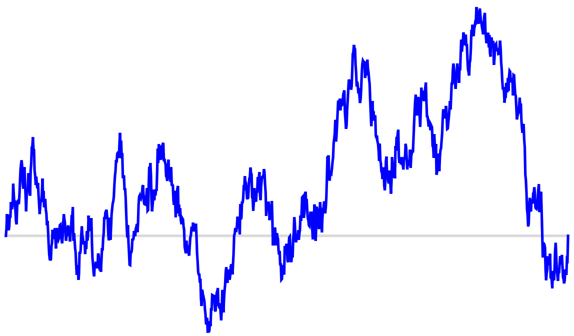
Short time spreading $\ell_{\|} \sim t^{2/3} \Rightarrow$ finite volume not felt away from boundaries

Late time saturation $\ell_{\|} \sim$ full system size $\Rightarrow$ boundary conditions

- Periodic $x \equiv x + 1$, simplest for computations
- Open $\partial_x h(0,t) = \sigma_a$ and $\partial_x h(1,t) = \sigma_b$ σ_a, $\sigma_b \leftrightsquigarrow$ densities of the reservoirs

Stationary state at the KPZ fixed point: statistics of $h_{\text{st}}(x) \underset{t\to\infty}{=} h(x,t) - h(0,t)$

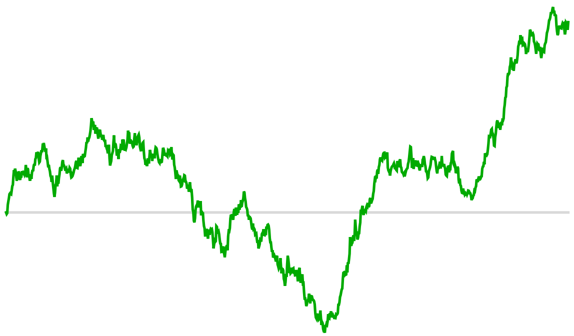
periodic



Brownian bridge

$\text{Var}(h_{\text{st}}(x)) = x(1-x)$

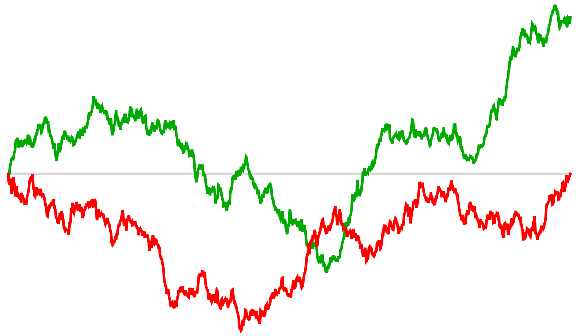
open $\sigma_a = \sigma_b = 0$



Brownian motion

$\langle h_{\text{st}}(x) \rangle = 0$

open $\sigma_a = -\infty$ $\sigma_b = \infty$



Brownian motion + Brownian excursion

$\langle h_{\text{st}}(x) \rangle \propto -\sqrt{x(1-x)}$

Stationary large deviations (periodic boundaries)

Stationary state $h_{\mathrm{st}}(x) \underset{t\to\infty}{=} h(x,t) - h(0,t)$ What about $h(x,t)$ alone ?

Typically $h(x,t) \simeq Jt$ and $h(x,t) - Jt \underset{t\to\infty}{\to}$ Gaussian distribution

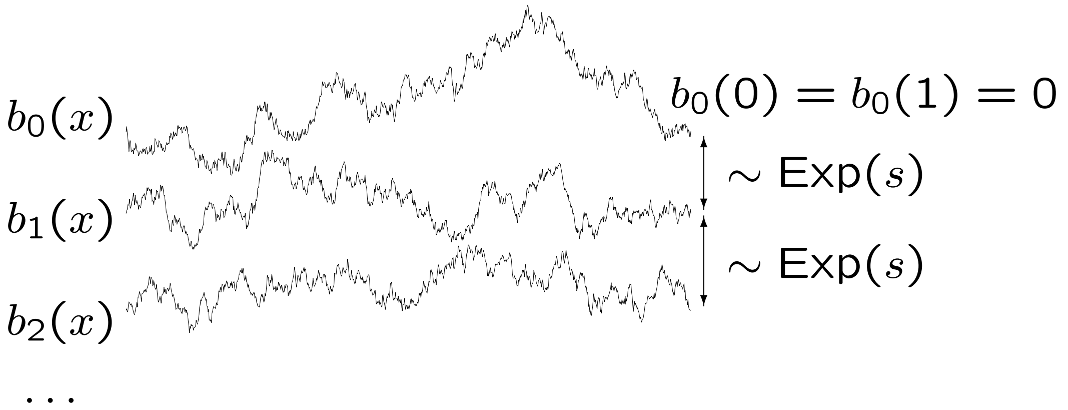
KPZ universality for large deviations of rare events $h(x,t) \simeq jt$, $j \neq J$

$$\langle \mathrm{e}^{sh(x,t)} \rangle \underset{t\to\infty}{\simeq} \theta(s)\, \mathrm{e}^{tF(s)} \text{ with } \begin{cases} F(s) = \chi(v) \\ s = \chi'(v) \end{cases} \text{ and } \chi(v) = -\frac{\mathrm{Li}_{5/2}(-\mathrm{e}^{v})}{\sqrt{2\pi}}$$

$$\mathrm{Li}_{5/2}(y) = \sum_{n=1}^{\infty} \frac{y^n}{n^{5/2}} \text{ for } |y| < 1$$ branch point at $y = 1 \Rightarrow$ Riemann surface

Dependency on the initial state $h_0(x)$
[Mallick-Prolhac 2018]

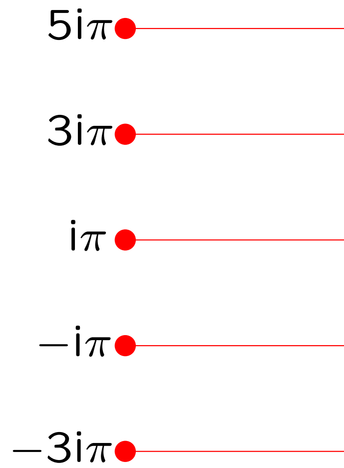
$$\theta(s) \propto \mathbb{P}(b_0 < h_0)$$



Exact formulas $\theta(s)$ simple initial states $\underset{\text{conjectures}}{\Rightarrow}$ flat case $h_0(x) = 0$

$$\mathbb{P}(b_0 < 0) = \frac{\exp(-\frac{1}{2}\int_{-\infty}^{v} \mathrm{d}u\, \chi''(u)^2)}{(1+\mathrm{e}^{v})^{1/4}}$$

Analytic continuation to higher eigenstates



$$\langle \mathrm{e}^{s h(x,t)} \rangle = \sum_n \theta_n(s)\, \mathrm{e}^{\mathrm{i} p_n x + t e_n(s)}$$

$$e_0(s) = \chi(v) \qquad s = \chi'(v) \qquad p_0 = 0$$

infinitely many branch points

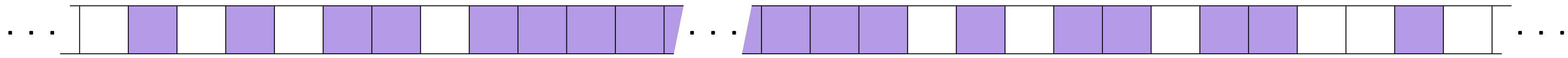
$$\chi'(v) = \sum_{a=\mathbb{Z}+1/2} \sqrt{4\mathrm{i}\pi a - 2v} - \infty$$

Analytic continuation sector $p_n = 0$

$$\chi'(v) \to \sum_{a=\mathbb{Z}+1/2} \sigma_a(P)\sqrt{4\mathrm{i}\pi a - 2v} \quad \text{with} \begin{cases} \sigma_a(P) = -1 & \text{if } a \in P \\ \sigma_a(P) = 1 & \text{if } a \notin P \end{cases}$$

Eigenstates $n = (P, H)$ with $\chi'(v) \to \chi_{P,H}(v) = \sum_{a=\mathbb{Z}+1/2} \frac{\sigma_a(P) + \sigma_a(H)}{2} \sqrt{4\mathrm{i}\pi a - 2v}$

independent excitations on each side of a Fermi sea [Prolhac 2014]



$$\underbrace{-\tfrac{5}{2} \;\; -\tfrac{1}{2}}_{P_-} \; \underbrace{\tfrac{1}{2} \;\; \tfrac{7}{2}}_{H_+} \qquad |P_-| = |H_+| \quad |P_+| = |H_-| \qquad \underbrace{-\tfrac{11}{2} \;\; -\tfrac{7}{2} \;\; -\tfrac{1}{2}}_{H_-} \; \underbrace{\tfrac{1}{2} \;\; \tfrac{3}{2} \;\; \tfrac{9}{2}}_{P_+}$$

$$\frac{p_n}{2\pi} = \sum_{a \in P} a - \sum_{a \in H} a$$

Riemann surface with connected components indexed by $(P \cup H) \setminus (P \cap H)$ [Prolhac 2020]

Ideal Fermi gas

$$Z_{\mathrm{GC}} = \prod_{j \in \mathbb{Z}} \Big(1 + \mathrm{e}^{\mu - \frac{1}{2}(\frac{2\pi j}{L})^2}\Big) = \sum_{P,H \subset \mathbb{Z}+1/2} (-1)^{|P|+|H|} \mathrm{e}^{L\chi'_{P,H}(\mu)} \underset{L\to\infty}{\simeq} \mathrm{e}^{L\chi'(\mu)}$$

Initial conditions

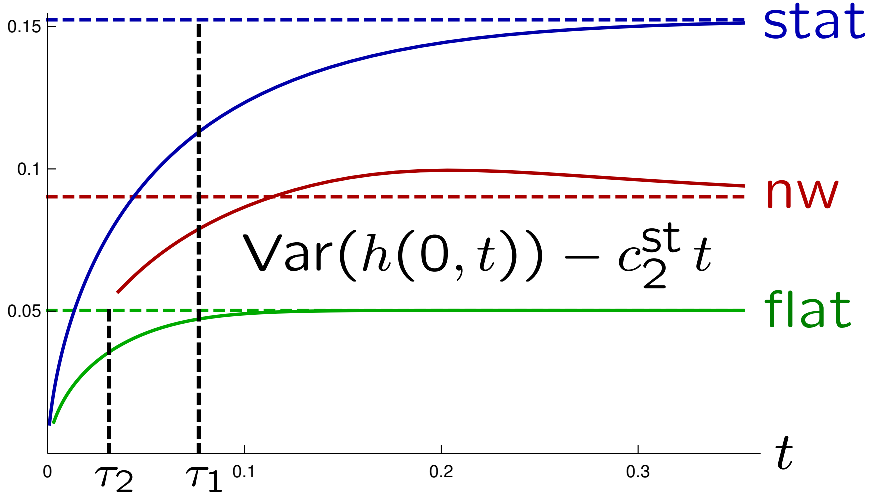
$$\langle \mathrm{e}^{sh(x,t)} \rangle = \sum_n \theta_n(s)\, \mathrm{e}^{\mathrm{i}p_n x + t e_n(s)} \qquad \chi'_{P,H}(v) = s \qquad D_{P,H}(v) \propto \mathrm{e}^{\frac{1}{2}\int_{-\infty}^{v} \mathrm{d}u\, \chi''_{P,H}(u)^2}$$

$$\theta^{\text{stat}}_{P,H}(s) = \frac{\sqrt{2\pi}\, s^2\, D_{P,H}(v)^2}{\mathrm{e}^v\, \chi''_{P,H}(v)}$$

$$\theta^{\text{flat}}_{P,H}(s) = 1_{\{P=H\}} \frac{s\, D_{P,H}(v)}{(1+\mathrm{e}^v)^{1/4}\, \chi''_{P,H}(v)}$$

$$\theta^{\text{nw}}_{P,H}(s) = \frac{s\, D_{P,H}(v)^2}{\chi''_{P,H}(v)}$$

[Prolhac 2015, 2016]

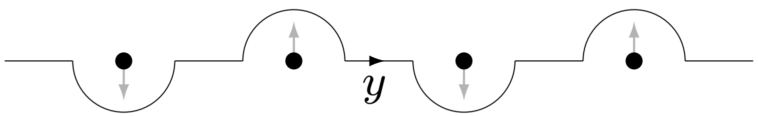


Strong Mpemba effect $\theta^{\text{flat}}_1(s) = 0 \Rightarrow$ fastest relaxation from flat initial state

Open boundaries   spectral gaps $e_n(s)$ known [Godreau-Prolhac 2020, 2021]

$$\left.\begin{array}{l} \partial_x h(0,t) = -\infty \\ \partial_x h(1,t) = +\infty \end{array}\right\} \Rightarrow \chi(v) = \frac{1}{6\pi} \int_{-\infty}^{\infty} \mathrm{d}y\, \frac{y^4(y^2-1)}{y^2 + \mathrm{e}^{y^2 - v}}$$

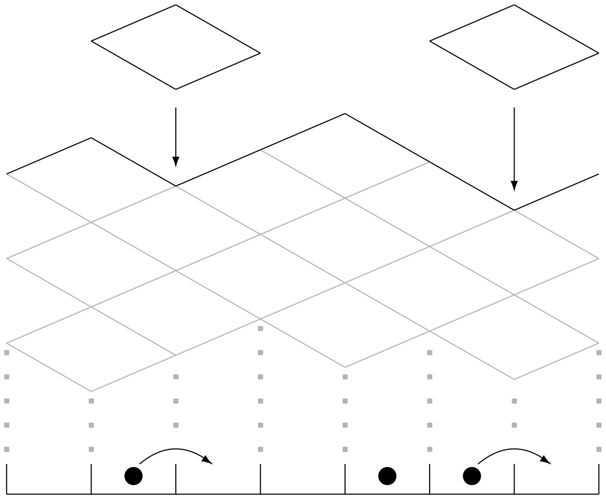
Analytic continuation



Pre-factors $\theta_n(s)$ for simple initial states ?

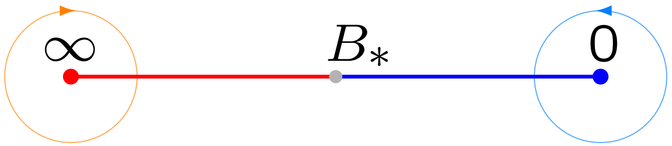
KPZ fluctuations in finite volume

$\Downarrow$ discretize



$N = 3 \qquad L = 7$

$\Downarrow$ diagonalize



$$B\, z_j^L = (z_j - 1)^N$$

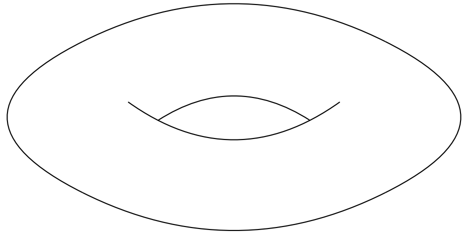
III Riemann surface approach

for

TASEP

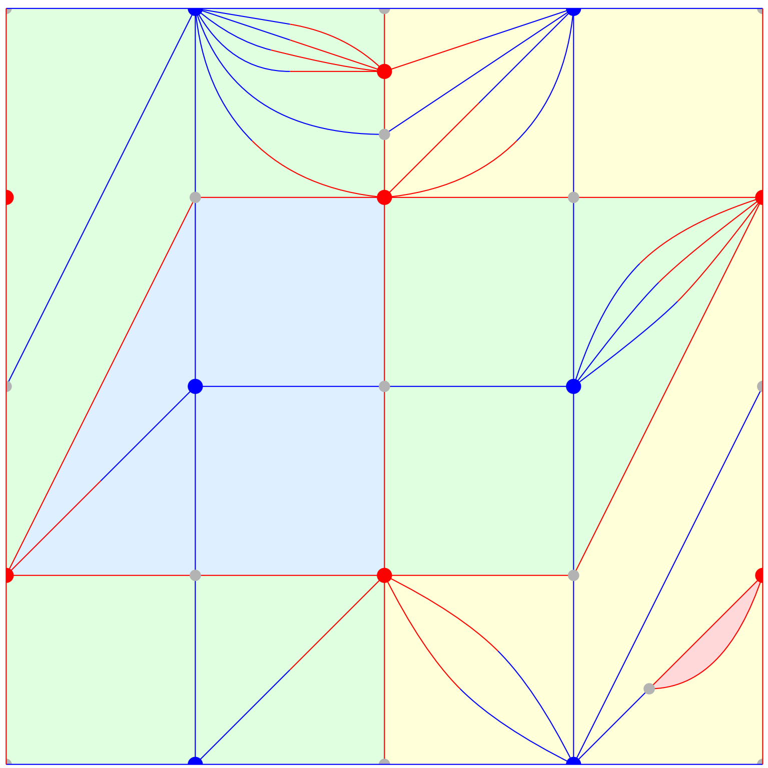
$\Rightarrow$ lift

$\Uparrow$ g $\sim \binom{L}{N} \to \infty$

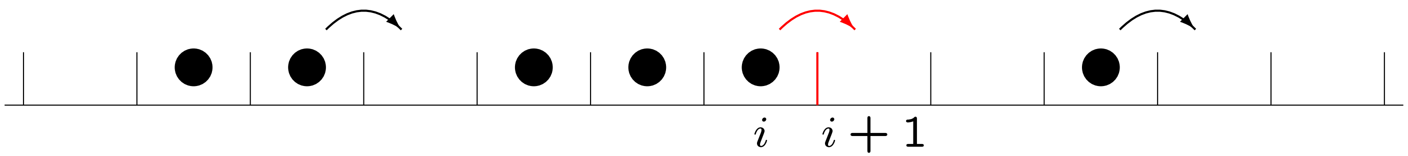


genus
g = 1

$\Uparrow$ glue



Master equation for TASEP



+ boundaries

Memoryless dynamics, depends only on current state $\Rightarrow$ Markov process

Master equation $\dfrac{\mathrm{d}}{\mathrm{d}t}P_t(\mathcal{C}) = \sum_{\mathcal{C}' \neq \mathcal{C}} \Big(w_{\mathcal{C} \leftarrow \mathcal{C}'}\, P_t(\mathcal{C}') - w_{\mathcal{C}' \leftarrow \mathcal{C}}\, P_t(\mathcal{C}) \Big) \quad \Rightarrow \quad |P_t\rangle = \mathrm{e}^{tM} |P_0\rangle$

No detailed balance $\Rightarrow$ Markov matrix M non-Hermitian

Current Q_t non-Markovian: $Q_t \to Q_t + 1$ when a particle moves from i to $i+1$

Deformed generator: $F_t(\mathcal{C}) = \sum_{Q \in \mathbb{Z}} g^Q\, P_t(\mathcal{C}, Q)$

$$\frac{\mathrm{d}}{\mathrm{d}t}F_t(\mathcal{C}) = \sum_{\mathcal{C}' \neq \mathcal{C}} \Big(g^{\delta Q_{\mathcal{C} \leftarrow \mathcal{C}'}}\, w_{\mathcal{C} \leftarrow \mathcal{C}'}\, F_t(\mathcal{C}') - w_{\mathcal{C}' \leftarrow \mathcal{C}}\, F_t(\mathcal{C}) \Big) \quad \Rightarrow \quad \Big\langle g^{Q_t} \Big\rangle = \sum_{\mathcal{C}} \langle \mathcal{C} | \mathrm{e}^{tM(g)} | P_0 \rangle$$

Expansion over the eigenstates of $M(g)$: complicated sum of algebraic functions

Probability of the current: contour integral on a Riemann surface

Spectral curve and Riemann surface

parameter-dependent $\Omega \times \Omega$ matrix

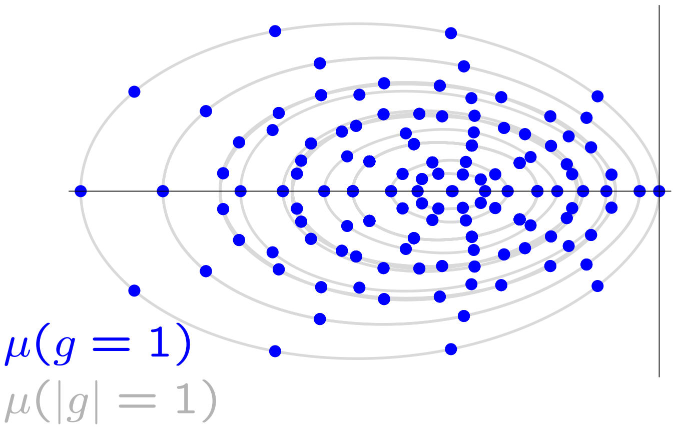
$M(g) \quad g \leftrightarrow Q_t$

$\Rightarrow$

spectral curve $\mathcal{S}: \det(\mu \, \mathrm{Id} - M(g)) = 0$

complex algebraic curve $\quad \mu, g \in \mathbb{C}$

Spectrum $M(g) \quad \Omega$ points

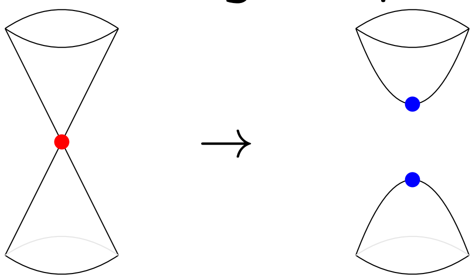


a point $(\mu, g) \in \mathcal{S}$	=	a specific eigenstate of $M(g)$ for some $g \in \mathbb{C}$

Example: stationary point $o \in \mathcal{S}$ with $\mu(o) = 0$, $g(o) = 1$

analytic continuation $\mu(g) \Rightarrow$ whole spectrum

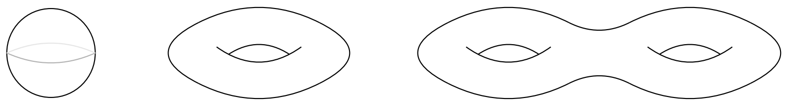
Remove singular points



Add points with $g = \infty$

$\Rightarrow$

$\mathcal{S} \rightarrow$ (compact) Riemann surface $\mathcal{R}$



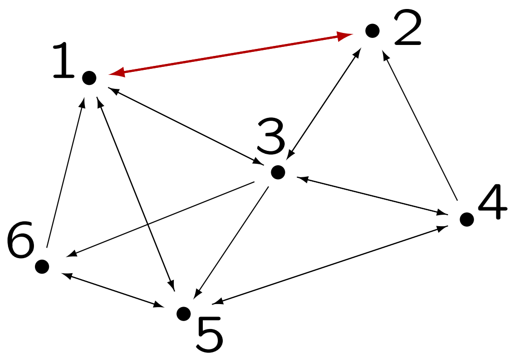
$+$

analytic structure (functions, differentials)

local parameter, residue theorem

Example: single current $\Rightarrow$ hyperelliptic $\mathcal{R}$ [Prolhac 2023]

Markov process Ω states

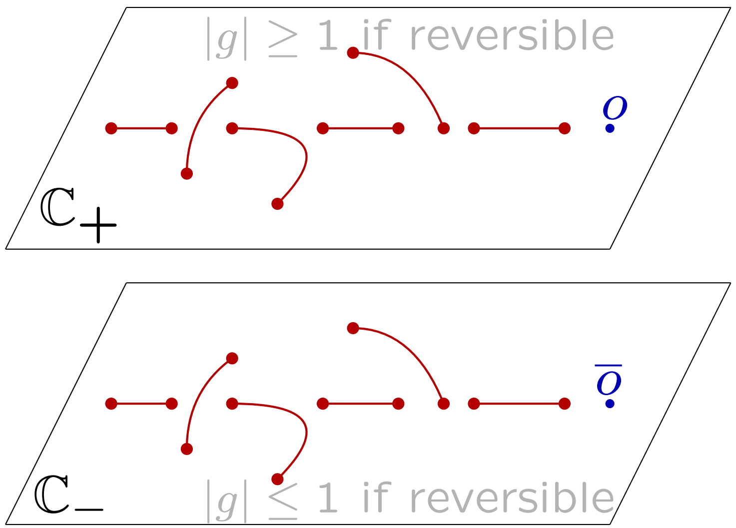


Deformed generator current Q_t $1 \leftrightarrow 2$

$$M(g) = \begin{pmatrix} -\ldots & g^{-1} w_{1\leftarrow 2} & w_{1\leftarrow 3} & \ldots \\ g\, w_{2\leftarrow 1} & -\ldots & w_{2\leftarrow 3} & \ldots \\ w_{3\leftarrow 1} & w_{3\leftarrow 2} & -\ldots & \ldots \\ \ldots & \ldots & \ldots & -\ldots \end{pmatrix}$$

$$\det(\mu\,\mathrm{Id} - M(g)) = P_0(\mu) + g\, P_+(\mu) + g^{-1} P_-(\mu)$$

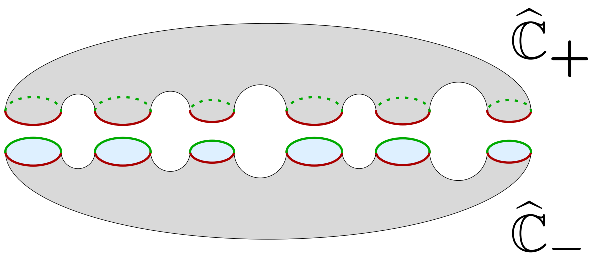
On the spectral curve $\det(\mu\,\mathrm{Id} - M(g)) = 0$ $\Rightarrow$ $g = \dfrac{-P_0(\mu) \pm \sqrt{\Delta(\mu)}}{2P_+(\mu)}$



Points $[\lambda, \pm] \in \mathcal{R}$

Ω branch cuts

genus $\mathrm{g} = \Omega - 1$



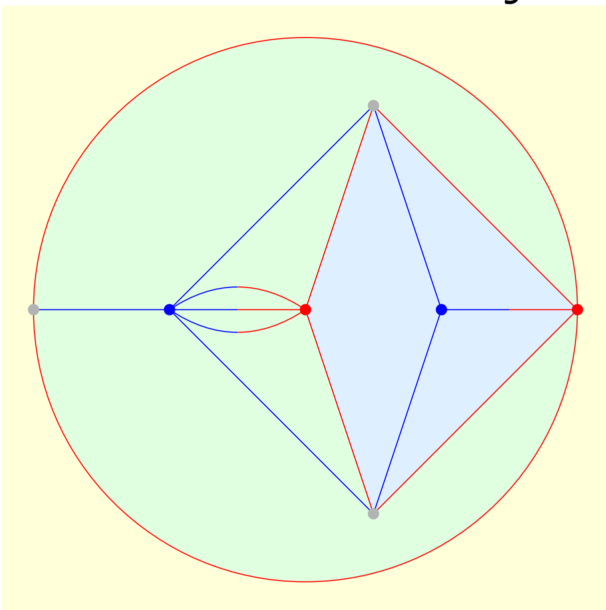
TASEP

Periodic boundaries $L = 5$ $N = 2$

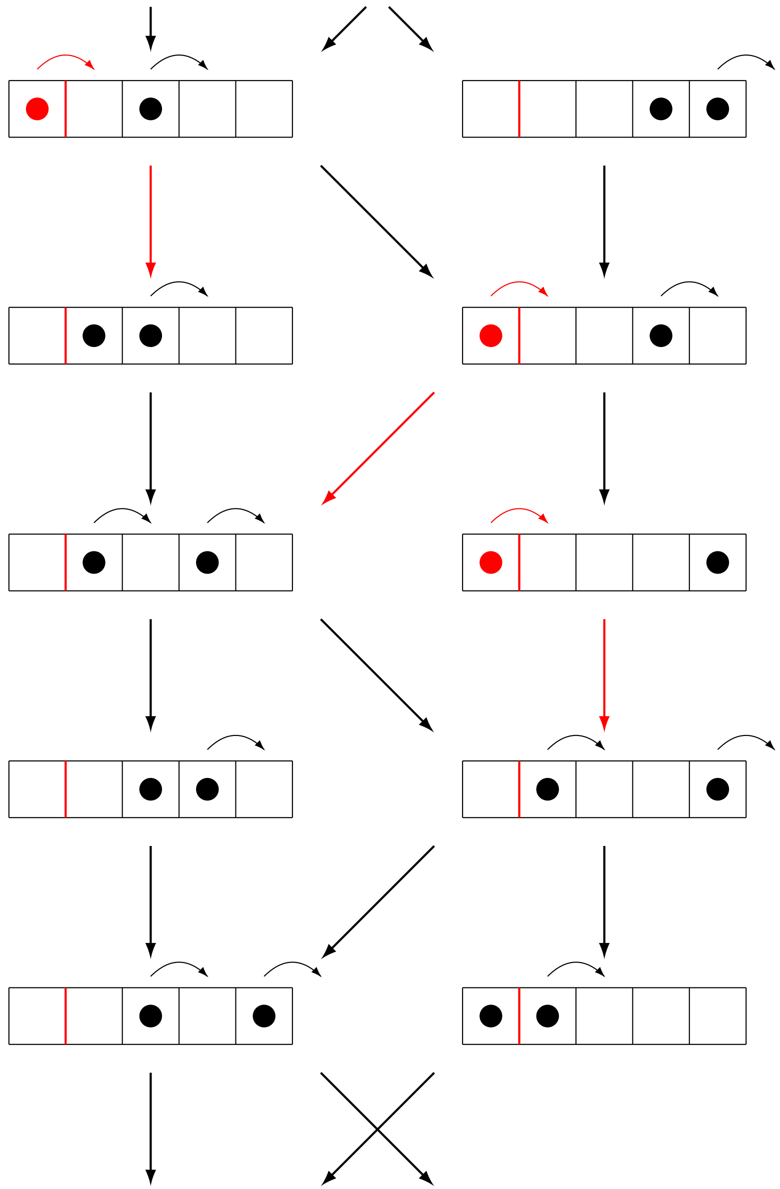
N particles on L sites
$\Omega = \binom{L}{N}$ possible states

Current Q_t through a single bond
- many transitions involved
- complicated ramification structure in the variable g for the Riemann surface $\mathcal{R}$
- highly singular algebraic curve $\mathcal{S}$

Integrability $\Rightarrow$ simple parametrization of $\mathcal{R}$
analytic continuation



$\leftrightsquigarrow$



Bethe ansatz for TASEP with periodic boundaries

Free particles (no exclusion): N independent random walks on L sites

$\Rightarrow$ eigenvector = product of N plane waves $\prod_{j=1}^{N} \mathrm{e}^{\mathrm{i}k_j x_j/L}$, $\frac{k_j}{2\pi} \in [\![1, L]\!]$

TASEP $\quad M(g) \sim H_{\mathsf{XXZ}}$ $\quad$ exclusion interaction integrable $\Rightarrow$ Bethe ansatz

$\Rightarrow$ eigenvector = linear combination of plane waves $\sum_{\sigma \in S_N} A_\sigma \prod_{j=1}^{N} \mathrm{e}^{\mathrm{i}k_j x_{\sigma(j)}/L}$

Periodic boundaries $\Rightarrow$ wave numbers k_j quantized by Bethe equations

$$z_j^L = \frac{(-1)^{N-1}}{g} \prod_{\ell=1}^{N} \frac{1 - z_j}{1 - z_\ell}$$

Bethe roots $z_j = \mathrm{e}^{\mathrm{i}k_j/L}/g^{1/L}$

Eigenvalue, overlaps explicit
symmetric functions of $z_1, \ldots, z_N$

$\Rightarrow$

Sheets of the Riemann surface $\mathcal{R}$
= branches of sets $\{z_1(g), \ldots, z_N(g)\}$

Good parametrization
Riemann surface $\mathcal{R}$

$$B = -g \prod_{\ell=1}^{N} (1 - z_\ell) \quad \Rightarrow \quad \boxed{B\, z_j^L = (z_j - 1)^N}$$

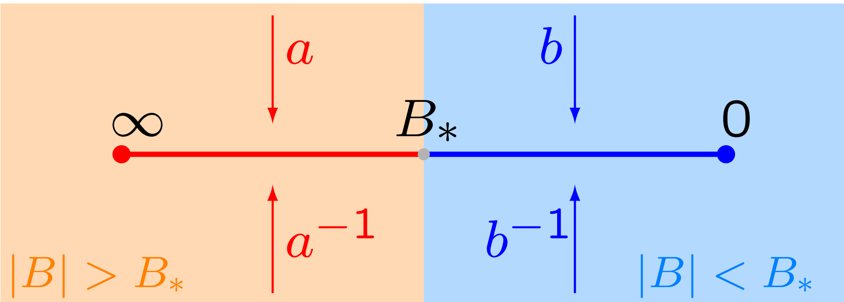
Bethe root functions $z_j(B)$

N distinct momenta $z_j(B)$ ("fermions")
solution of $B\, z_j^L = (z_j - 1)^N$

L solutions $z_j(B)$ analytic in $\mathbb{C} \setminus \mathbb{R}^-$
with branch points 0, $B_* > 0$, ∞

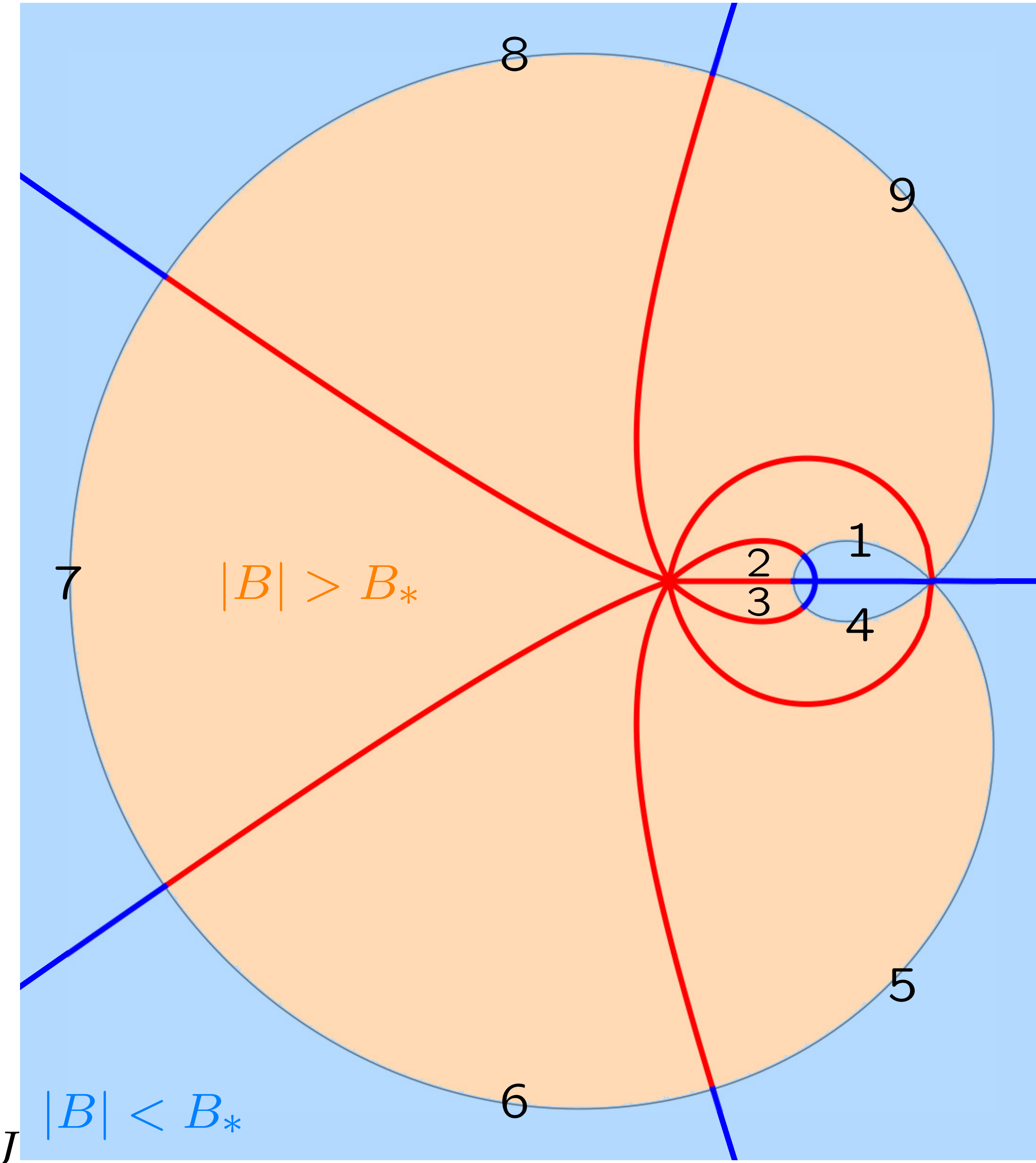
Analytic continuations $y_j \to y_k$
ramification $\leftrightsquigarrow$ cyclic permutations

$|B| > B_*$ $a = (1, \ldots, L)$
$|B| < B_*$ $b = (1, \ldots, N)(N+1, \ldots, L)$



[Prolhac 2020]

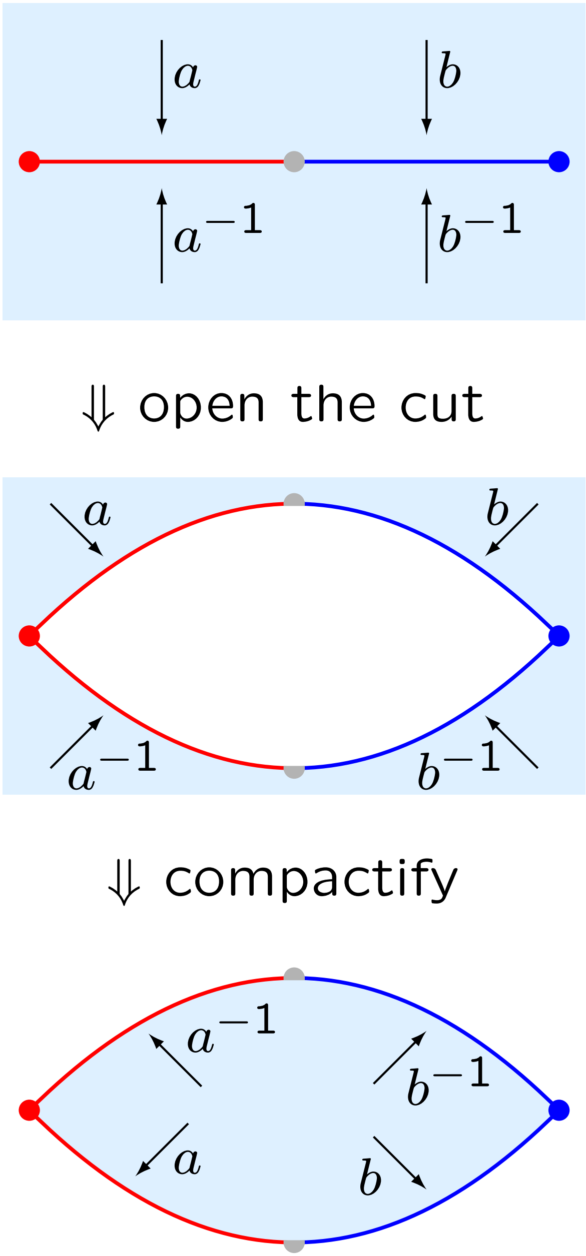
Sheets of $\mathcal{R}$ indexed by sets
$J = \{j_1, \ldots, j_N\} \subset [\![1, L]\!] \quad \Rightarrow \quad z_j(B),\ j \in J$

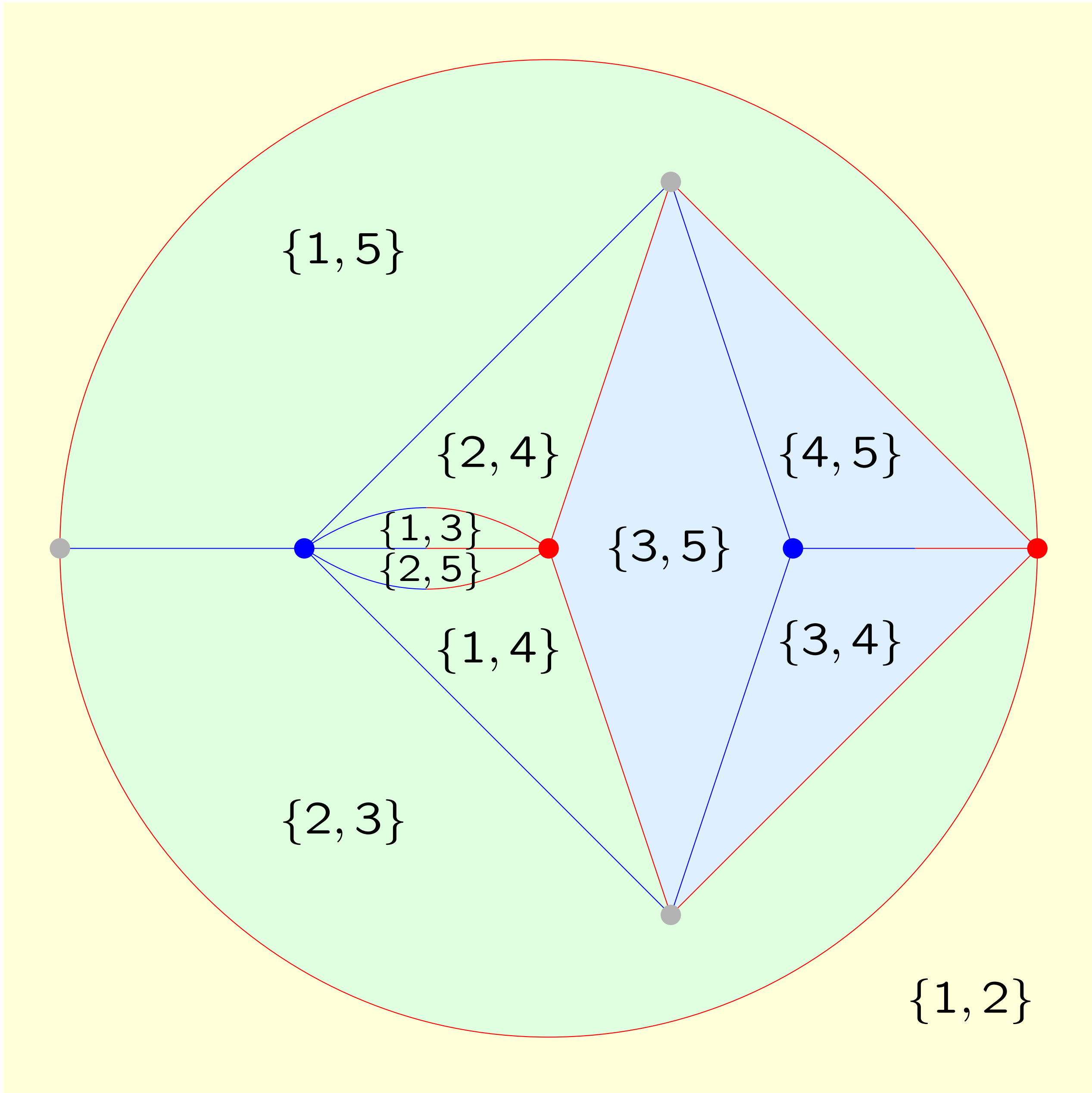


Domains $z_j(\mathbb{C})$ for $L = 9$ $N = 4$

Glue sheets together $L = 5,\ N = 2$

$a = (1,2,3,4,5)$
$b = (1,2)(3,4,5)$



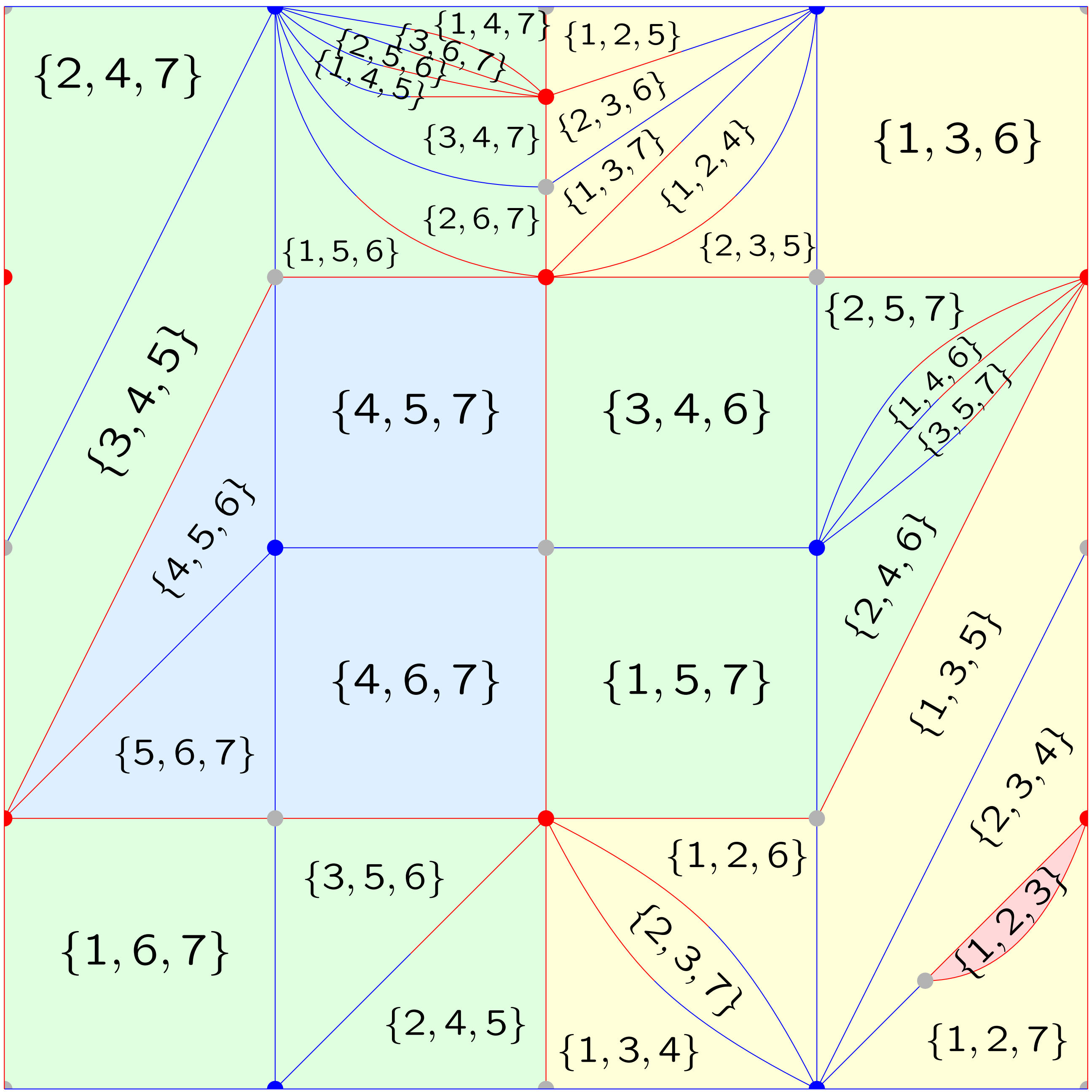


planar graph $\Rightarrow$ $\mathcal{R} \equiv \hat{\mathbb{C}}$ Riemann sphere

Glue sheets together $L = 7$, $N = 3$

$a = (1,2,3,4,5,6,7)$

$b = (1,2,3)(4,5,6,7)$



non planar graph
(opposite sides
glued together)

$\Downarrow$

$\mathrm{g} = 1$

$\mathcal{R}$ is a torus

General L, N

permutations a and b

$\Downarrow$

global topology of $\mathcal{R}$

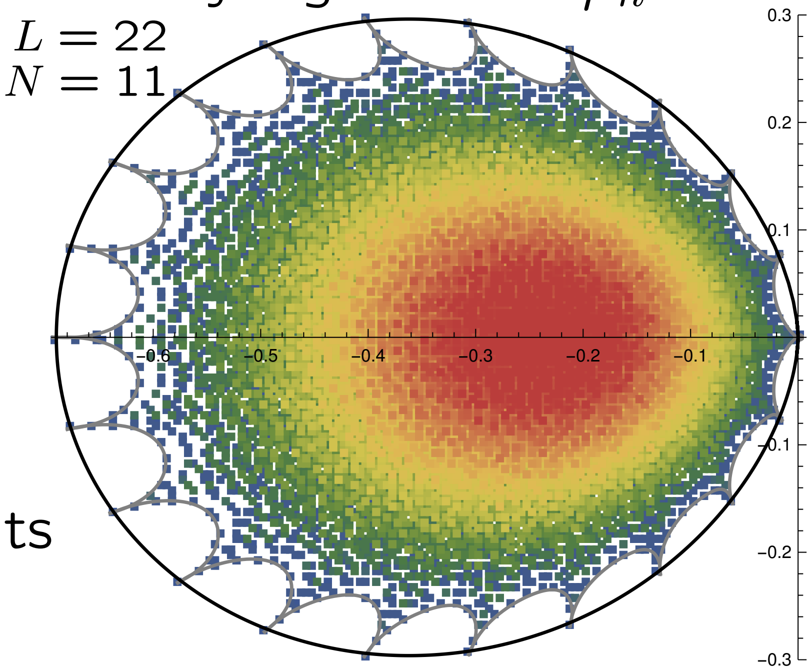
Euler characteristic
$2 - 2\mathrm{g} = V - E + F$
for any tiling of $\mathcal{R}$

Spectrum of the Markov generator M of TASEP

Eigenvalue $\mu = \sum_{j=1}^{N} \left(\frac{1}{z_j} - 1 \right)$

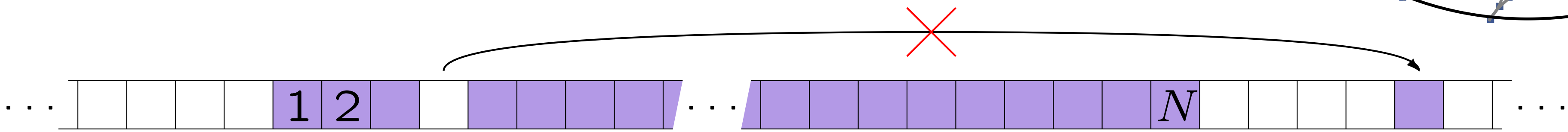
dynamics $e^{t\mu}$
$\mathrm{Re}\,\mu < 0$

[Prolhac 2013]
Density eigenvalues μ_n of M

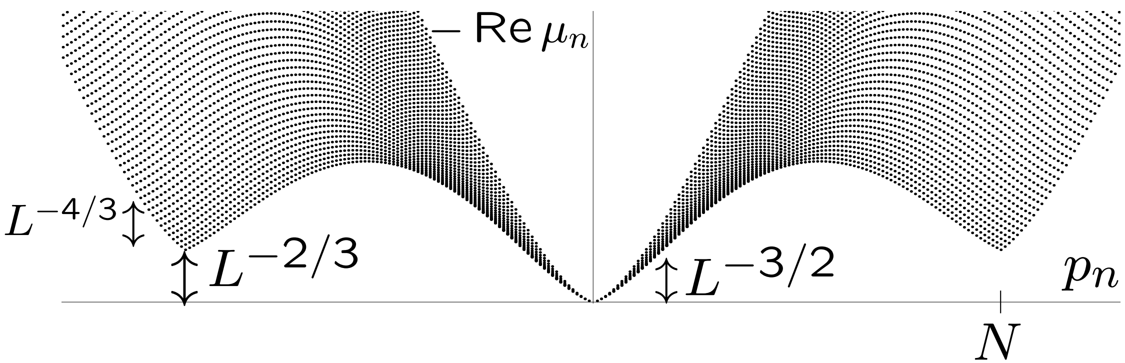


KPZ time scale $t \sim L^{3/2}$
$\Rightarrow$ $\mathrm{Re}\,\mu \sim L^{-3/2}$ rightmost peak
$\Rightarrow$ particle-hole excitations at both edges of the Fermi sea $J = [\![1, N]\!]$

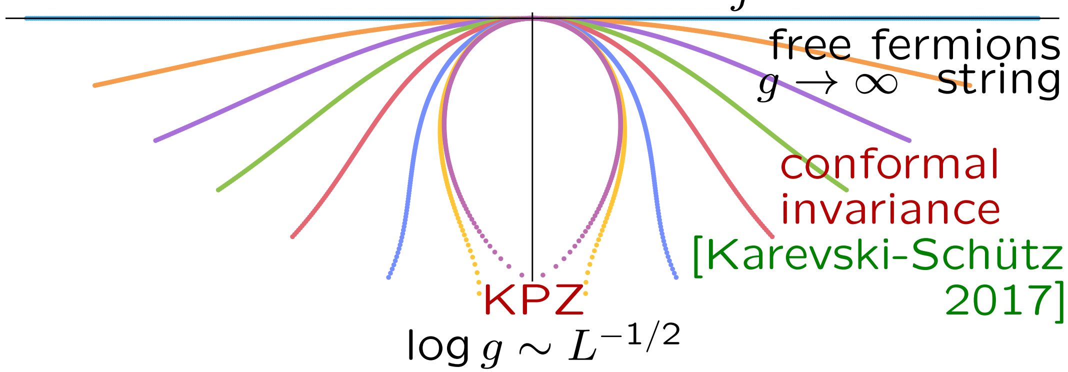
Analogous Luttinger liquid, but different exponents
$\Rightarrow$ Umklapp excitations $\mathrm{Re}\,\mu \sim L^{-2/3}$ suppressed



One particle-hole excitations



Momenta $\log z_j$



Probability of an integer counting process Q_t [Prolhac 2022]

Markov integer counting process

$$\left\langle g^{Q_t}\right\rangle = \sum_{Q\in\mathbb{Z}} \mathbb{P}(Q_t = Q)\, g^Q \quad \Leftrightarrow \quad \mathbb{P}(Q_t = Q) = \oint_\gamma \frac{\mathrm{d}g}{g^{Q+1}} \left\langle g^{Q_t}\right\rangle$$

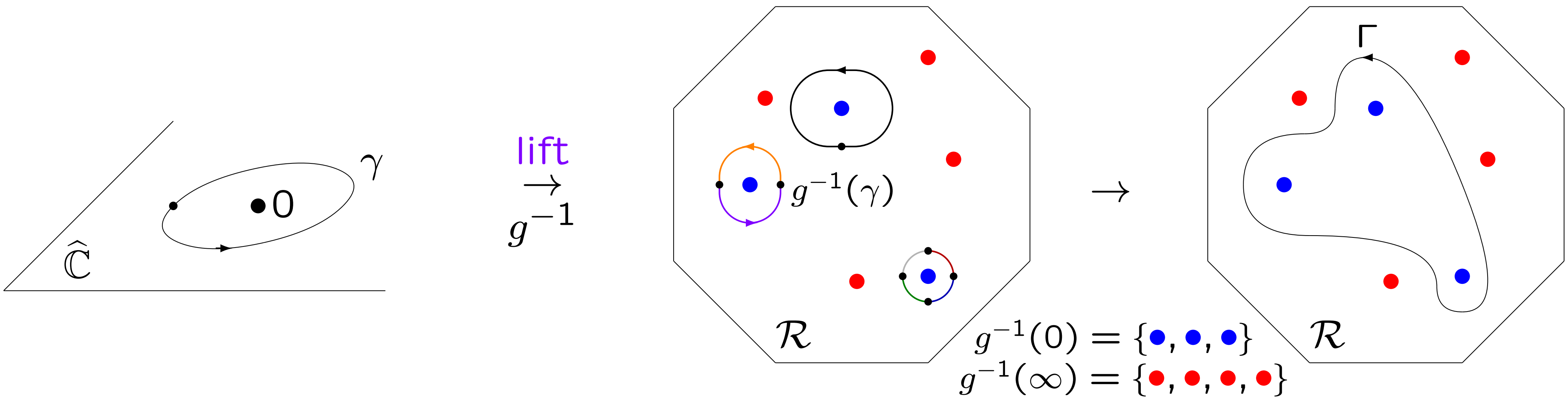
$$\left\langle g^{Q_t}\right\rangle = \sum_{\mathcal{C}} \langle\mathcal{C}|\mathrm{e}^{tM(g)}|P_0\rangle$$

expansion over eigenstates

$$\Rightarrow \quad \mathbb{P}(Q_t = Q) = \oint_\gamma \frac{\mathrm{d}g}{g^{Q+1}} \sum_{n=1}^{\Omega} \underbrace{\frac{\sum_{\mathcal{C}}\langle\mathcal{C}|\psi_n\rangle\langle\psi_n|P_0\rangle}{\langle\psi_n|\psi_n\rangle}}\, \mathrm{e}^{t\mu_n}$$

$$M(g)|\psi_n(g)\rangle = \mu_n(g)|\psi_n(g)\rangle \quad \Rightarrow \quad |\psi_n\rangle = |\psi(\mu_n, g)\rangle \quad \Rightarrow$$

meromorphic function $\mathcal{N}$ of $p = (\mu_n, g) \in \mathcal{R}$



$$\mathbb{P}(Q_t = Q) = \oint_{p\in\Gamma} \frac{\mathrm{d}g(p)}{g(p)^{Q+1}}\, \mathcal{N}(p)\, \mathrm{e}^{t\mu(p)}$$

Probability TASEP height H_t / current Q_t [Prolhac 2020]

$$\mathbb{P}(Q_t = Q) = \oint_{p\in\Gamma} \frac{\mathrm{d}g(p)}{g(p)^{Q+1}} \mathcal{N}(p)\, \mathrm{e}^{t\mu(p)} \;\Rightarrow\; \mathbb{P}(H_{i,t} \geq H) = \oint_{p\in\Gamma} \frac{\mathrm{d}B}{2\mathrm{i}\pi B}\, \mathrm{e}^{\int_o^p \left(t\,\mathrm{d}\mu - H\,\frac{\mathrm{d}g}{g} + \omega\right)}$$

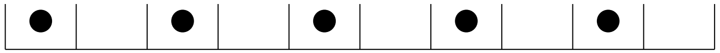
$$H \in H_{i,0} + \mathbb{Z}$$

Bethe ansatz $\Rightarrow$ exact formulas for the meromorphic differential ω

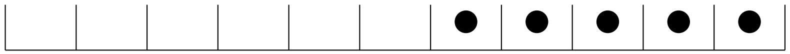
$$\omega_{\text{stat}} = \Big(\frac{N(L-N)}{L}\,\kappa^2 + \frac{\kappa}{1-g^{-1}} - 1\Big)\frac{\mathrm{d}B}{B}$$

$\mathcal{C}_0$ random, uniformly on Ω states

$$\omega_{\text{flat}} = \Big(\frac{L}{8}\,\kappa^2 + \frac{\kappa}{2} - \frac{1/4}{1+2^{-L}B^{-1}}\Big)\frac{\mathrm{d}B}{B}$$

$\mathcal{C}_0 =$ 

$$\omega_{\text{dw}} = \frac{N(L-N)}{L}\,\kappa^2\,\frac{\mathrm{d}B}{B}$$

$\mathcal{C}_0 =$ 

$$\kappa = \frac{\mathrm{d}\log g}{\mathrm{d}\log B} = \frac{L}{N}\sum_j \frac{1-z_j}{L-(L-N)z_j}$$

poles ramified for B

zeroes: eigenstates of $M(g(p))$ coincide

(Kato's exceptional points)

Intrinsic characterization of space of differentials ω on $\mathcal{R}$ for all initial states ?

KPZ scaling $\quad B/B_* = -\mathrm{e}^{v} \Rightarrow \kappa(p) \simeq \frac{\chi''_{P,H}(v)}{\sqrt{\rho(1-\rho)L}}$

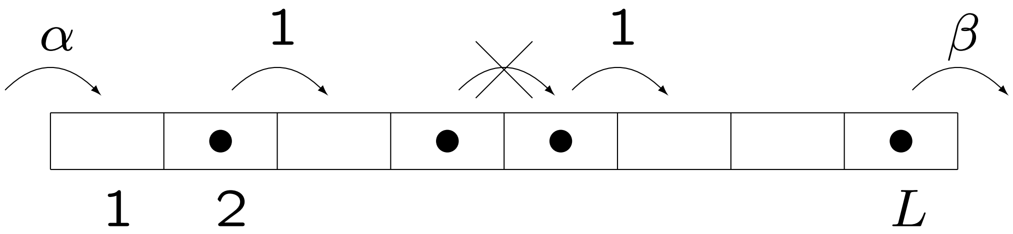
half-integer polylogarithm

$B = B_* \leftrightarrow v \in 2\mathrm{i}\pi(\mathbb{Z}+1/2)$

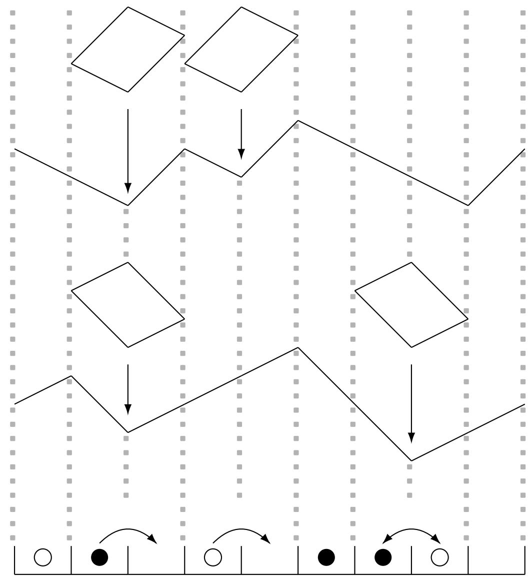
Open questions for KPZ fluctuations in finite volume

Open boundaries $\frac{B\,z^{2L+2}}{(z-1)^{L+2}} = \frac{(1-\alpha z)(1-\overline{\alpha}z)(1-\beta z)(1-\overline{\beta}z)}{(2-z)^2}$

more branch points for B (up to 5)



$\overline{\alpha} = 1 - \alpha$
$\overline{\beta} = 1 - \beta$



Several conserved quantities higher dimensional complex manifold

$$B\,z_j^L = (z_j - 1)^N \prod_{\ell=1}^{M} (w_\ell - z_j) \qquad j = 1, \ldots, M+N$$

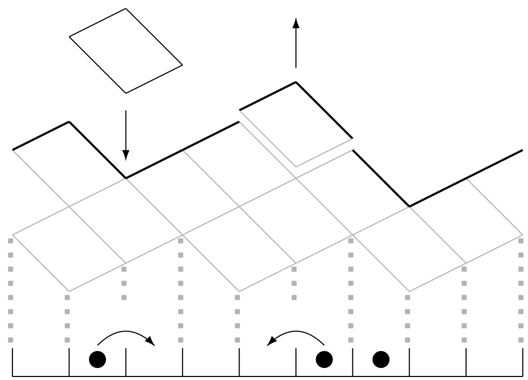
$$C \prod_{k=1}^{M+N} (z_k - w_i) = (w_i - 1)^M \qquad i = 1, \ldots, M$$

KPZ equation renormalization group flow
equilibrium $\rightarrow$ KPZ fixed point

$$g\left(\frac{1-y_j}{1-qy_j}\right)^L + (-1)^N \prod_{k=1}^{N} \frac{y_j - qy_k}{y_k - qy_j} = 0$$

fully coupled Bethe equations

absence of nice global parametrization B of $\mathcal{R}$?

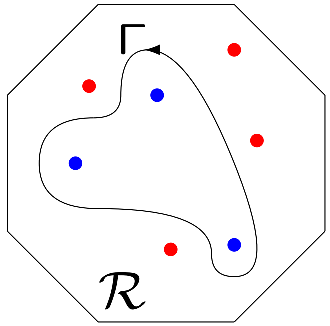


Conclusions

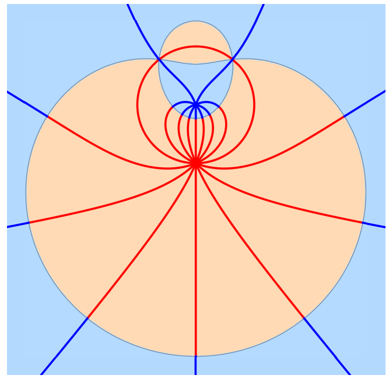
Tools from algebraic geometry for statistics of current-like observables Q_t

- stationary large deviations $\rightarrow$ relaxation times by analytic continuation
- time-dependent statistics from contour integral on Riemann surface $\mathcal{R}$

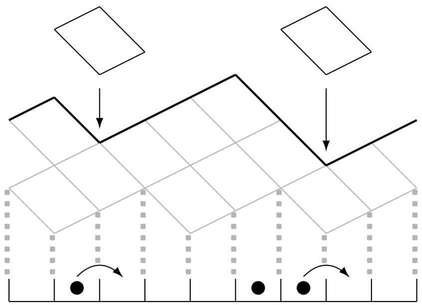
$$\mathbb{P}(Q_t = Q) = \oint_{p\in\Gamma} \frac{\mathrm{d}g(p)}{g(p)^{Q+1}} \mathcal{N}(p)\, \mathrm{e}^{t\mu(p)}$$



- integrability $\Rightarrow$ exact ramification structure from Bethe ansatz

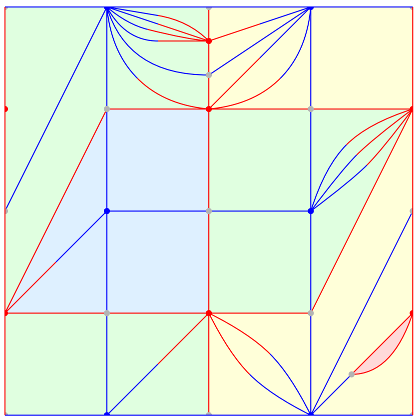


Prominent example: KPZ fluctuations in finite volume



TASEP

$\leftrightsquigarrow$



$t \sim L^{3/2}$
$\Rightarrow$
genus $\rightarrow \infty$

$\mathcal{R}_{\mathrm{KPZ}} \leftrightsquigarrow \mathrm{Li}_{3/2}$
infinite sum $\sqrt{}$