6 Field-Theoretical Methods in Quantum Magnetism

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Abstract. We present a review of different field theory techniques that have proved very useful in the study of quantum magnets in low dimensions. We first review the application of the spin-wave analysis and non-linear σ-model techniques in one and two dimensional quantum antiferromagnets. We discuss in particular the emergence of Haldane's conjecture for spin chains and ladders within this formalism. We also present a brief discussion on the non-linear σ-model description for the two-dimensional antiferromagnet in the square lattice. In a second part we review the method of abelian bosonization and its application to the study of the XXZ spin 1/2 chain and its generalizations, such as the dimerized chain. Non-abelian bosonization is used to describe both $SU(2)$ symmetric chains with arbitrary spin S and 2 leg ladders, rederiving Haldane's conjecture within this formalism. The inclusion of charge degrees of freedom leading to a Hubbard or a $t-J$ model is also discussed. Finally, we apply the abelian bosonization approach to the study of N-leg ladders in amagnetic field, which leads to a further extension of Haldane's conjecture.

6.1 Introduction

Field theory techniques have proven in the last decades to be a powerful tool in the understanding of quantum magnetism. One of its main interests lie in the relatively simple and universal description it can provide in studying condensed matter system, and in particular all the exotic behaviors that can be found in low dimensional strongly correlated systems. For example, phenomena like fractional excitations or spin-charge separation, which are going to be presented in this chapter, are some of the topics that find a very natural description in the field theory context. This approach has indeed allowed to understand the experimental data reflecting the presence of such unusual behaviors. Another important and more pragmatic issue is the fact that, once a field theory is built for describing a particular model, physical quantities such as correlation functions, the magnetic susceptibility or the specific heat can in general be easily computed. Moreover, the effect of microscopic modifications of the system, as well as the specificity of low dimensional systems, can also be simply understood through this approach.

The field theory approach have to be considered as a fundamental tool within the different techniques that are currently used to study condensed

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matter systems. It is complementary and in close connection to other techniques such as integrable models and numerical methods. As we show in this chapter, the Bethe Ansatz solution of the XXZ spin chain provides information about the system that is then used to construct the precise field theory model that describes its large scale behavior. The knowledge of the field theory allows to compute, for example, the behavior at large distances of correlation function in a much simpler way than with integrable model techniques. Finite size scaling analysis is another subject in which field theory have proven to be very useful, providing a natural link with numerical techniques also commonly used in condensed matter physics.

This chapter provides a review of field theory techniques that are used in the study of quantum magnets in low dimensions. In the first part we provide an overview of the spin-wave analysis in one and two dimensional quantum antiferromagnets. We then concentrate on the derivation of the non-linear σ-model that describes the low energy dynamics of spin S chains within the large S approach. The behavior of this model with and without a topological term, giving rise to Haldane's conjecture, is discussed. The results obtained in the context of spin-wave analysis and the non-linear σ-model are then generalized to the case of spin ladders. A brief discussion on the applications of this description for the two-dimensional antiferromagnet in the square lattice closes the first part of this chapter.

In the second part we review the method of abelian bosonization and apply it to the study of the XXZ spin $1/2$ chain in the presence of a magnetic field, which leads to the Luttinger liquid picture. We discuss how the microscopic data of the lattice model are related to the field theory parameters. We analyze in detail the computation of thermodynamic quantities and correlation functions within the bosonization method as well as certain modifications of the XXZ chain. The particularities and non-abelian bosonization description of the $SU(2)$ Heisenberg point are also discussed. We then briefly treat the generalization of those results to the Hubbard and $t-J$ models to illustrate the inclusion of charge degrees of freedom. Non-abelian bosonization is also applied to the study of the two leg $S = 1/2$ $SU(2)$ symmetric spin ladder and to higher spin one-dimensional chains where it is used to rederive Haldane's conjecture. Finally, we apply the abelian bosonization approach to the study of N-leg ladders, which leads to a further extension of Haldane's conjecture.

The general overview presented here aims not only at providing a description of the usual tools used in field theory for condensed matter physics, but also to show to the reader how this approach is in almost symbiotic connection with the other areas described in this book. We also mention in this chapter many topics that to date are still open questions with the hope that future progresses in field theory will help to elucidate those issues.

6.2 Path Integral for Spin Systems

Let us assume that we have an arbitrary number of spins $\{\mathbf{S}_n\}$ labeled by the generic site index n without, for the moment, making any further supposition about the geometry and dimensionality of such an array of spins. These operators satisfy an $SU(2)$ algebra on each site

$$[S_n^x, S_n^y] = iS_n^z \quad \text{and} \quad \mathbf{S}_n^2 = S(S+1), \tag{6.1}$$

We assume also that the system has a Hamiltonian $H(\{\mathbf{S}_i\})$ that we do neither need to specify for the moment. The idea is to define a path integral for such a system as proposed by Haldane (see for example [1], [2]). To avoid making heavier the notation, let us assume that we have first a single spin. Following [2], in the $2S+1$ dimensional Hilbert space, we define the states:

$$|\mathbf{n}\rangle = e^{i\theta(\hat{z}\times\mathbf{n})\cdot\mathbf{S}}\ |S,S\rangle \tag{6.2}$$

where $\mathbf{n}$ is a unit vector forming an angle θ with the quantization axis (z) and $|S,S\rangle$ is the highest weight state. A straightforward calculation shows that:

$$\langle\mathbf{n}|\mathbf{S}|\mathbf{n}\rangle = S\mathbf{n}. \tag{6.3}$$

One can also show that the internal product of two such states gives:

$$\langle\mathbf{n}_1|\mathbf{n}_2\rangle = e^{iS\Phi(\mathbf{n}_1,\mathbf{n}_2,\hat{z})}\left(\frac{1+\mathbf{n}_1\cdot\mathbf{n}_2}{2}\right)^S \tag{6.4}$$

where $\Phi(\mathbf{n}_1,\mathbf{n}_2,\hat{z})$ is the solid angle viewed from the origin formed by the triangle with vertices in $\mathbf{n}_1$, $\mathbf{n}_2$ and $\hat{z}$. Note that $\Phi(\mathbf{n}_1,\mathbf{n}_2,\hat{z})$, as a solid angle, is defined modulo 4π. This ambiguity has however no importance in (6.4) because of the periodicity of the exponential. With this over-complete basis, one can also write the identity operator in the Hilbert space:

$$I = \int \left(\frac{2S+1}{4\pi}\right) d^3n\ \delta(\mathbf{n}^2-1)|\mathbf{n}\rangle\langle\mathbf{n}| \tag{6.5}$$

which can be obtained by using the properties of the rotation matrices

$$D^S_{M,M'}(\mathbf{n}) = \langle S,M|e^{i\theta(\hat{z}\times\mathbf{n})\cdot\mathbf{S}}|S,M'\rangle$$

in the spins S representation :

$$\frac{2S+1}{4\pi}\int d^3n\ \delta(\mathbf{n}^2-1){D^{S}}^*_{M,M'}(\mathbf{n})D^S_{N,N'}(\mathbf{n}) = \delta_{M,N}\delta_{N',M'}.$$

Imagine now that we want to compute the partition function

$$Z = Tr\{e^{-\beta H}\}$$

seen as the evolution of the system in imaginary time with periodic boundary conditions. We can decompose the evolution in imaginary time into N infinitesimal steps of length δt, with $N \to \infty$, $N\delta t = \beta$. using then the Trotter formula:

$$Z = \lim_{N\to\infty} \left(e^{-\delta t H}\right)^N$$

an inserting an identity at each intermediary step, we obtain:

$$Z = \lim_{N\to\infty} \left(\frac{2S+1}{4\pi}\right)^N \left(\prod_{a=1}^{N} \int d^3 n_a \; \delta(\mathbf{n}_a^2 - 1)\langle \mathbf{n}(t_a)|e^{-\delta t H}|\mathbf{n}(t_{a+1})\rangle\right) .$$

If now, as in the standard path integral construction, we only keep in each infinitesimal step the first order in δt:

$$\langle \mathbf{n}(t_a)|e^{-\delta t H}|\mathbf{n}(t_{a+1})\rangle = \left[\langle \mathbf{n}(t_a)|\mathbf{n}(t_{a+1})\rangle + \delta t \langle \mathbf{n}(t_a)|H|\mathbf{n}(t_a)\rangle + O(\delta t^2)\right]$$

and we formally define the path integral measure:

$$\int \mathcal{D}\mathbf{n} \;=\; \lim_{N\to\infty} \left(\frac{2S+1}{4\pi}\right)^N \left(\prod_{a=1}^{N} \int d^3 n_a \; \delta(\mathbf{n}_a^2 - 1)\right)$$

using (6.4) we can write the partition function as:

$$Z = \int \mathcal{D}\mathbf{n} \; e^{-S[\mathbf{n}]} \tag{6.6}$$

with

$$\begin{aligned} S[\mathbf{n}] = &-iS \sum_a \; \Phi(\mathbf{n}(t_a), \mathbf{n}(t_{a+1}), \hat{z}) \\ &- S \sum_a \log\left(\frac{1 + \mathbf{n}(t_a)\cdot\mathbf{n}(t_{a+1})}{2}\right) + \delta t \sum_a \langle \mathbf{n}(t_a)|H|\mathbf{n}(t_a)\rangle . \end{aligned} \tag{6.7}$$

Since in the computation of the partition function we used periodic boundary conditions, namely: $\mathbf{n}(0) = \mathbf{n}(\beta)$, and if we suppose that the path described by $\mathbf{n}(t)$ is smooth [3], we see that $\sum_a \Phi(\mathbf{n}(t_a), \mathbf{n}(t_{a+1}), \hat{z})$ describes the solid angle, or the area in the unit sphere bounded by the curve $\mathbf{n}(t)$, $\mathcal{A}\{\mathbf{n}(t)\}$. As before, the independence of the partition function of the choice of the quantization axis or the ambiguity in the definition of the solid angle is a consequence of the 4π invariance of the phase factor. In (6.7), the second term is of order $(\delta t)^2$, the imaginary time continuum limit of this action is then :

$$S[\mathbf{n}] = -iS\mathcal{A}\{\mathbf{n}(t)\} + \int_0^\beta dt \langle \mathbf{n}(t)|H|\mathbf{n}(t)\rangle . \tag{6.8}$$

[3] This assumption is actually delicate, as in the standard Feynmann path integral, see [2] and references therein, but we ignore such technical details here.

Equation (6.8) is the main result of the path integral description for magnetic systems which we can now apply to spins chains, ladders and two-dimensional antiferromagnets.

6.3 Effective Action for Antiferromagnetic Spins Chains

Let us assume now that we have a collection of spins $\mathbf{S}_i^2 = S(S+1)$ forming a one dimensional array (chain) with the Hamiltonian:

$$H = J \sum_k \mathbf{S}_n \cdot \mathbf{S}_{n+1} \tag{6.9}$$

with $J > 0$. (6.9) is just the one-dimensional Heisenberg antiferromagnet. The action (6.8) takes then the explicit form:

$$S[\{\mathbf{n}_n\}] = -iS \sum_n \mathcal{A}\{\mathbf{n}_n(t)\} + JS^2 \int_0^\beta dt \sum_n \mathbf{n}_n(t) \cdot \mathbf{n}_{n+1}(t). \tag{6.10}$$

In order to take the continuum limit in the spatial direction, we need to identify the low energy, large scale degrees of freedom that can be considered as slowly varying fields in the action. We can, for this, make use of the known results from spin wave theory from which we know that low energy modes are found at zero and π momenta (see also below for the generalization to the case of ladders). We can then write the ansatz

$$\mathbf{n}_n = (-1)^n \sqrt{1 - a^2 \mathbf{l}_n^2}\; \mathbf{m}_n + a \mathbf{l}_n \tag{6.11}$$

with a the lattice spacing and $\mathbf{m}_n^2 = 1$. This result, which is valid for large S, is just telling us that the large scales behavior of the system is governed by fields representing a staggered and a quasi-homogeneous variation of the magnetization. The latter field, playing the rôle of angular momentum for $\mathbf{n}$ is chosen to have dimension of density and is responsible of a net magnetization which is supposed to be small. To order a^2, the relation $\mathbf{n}_n^2 = 1$ is equivalent to $\mathbf{m}_n \cdot \mathbf{l}_n = 0$. We can now introduce this form for the field $\mathbf{n}_n$ in (6.10) and keep only the lowest order in a to take the continuum limit. For the area term, by noticing that $\mathcal{A}\{-\mathbf{n}(t)\} = -\mathcal{A}\{\mathbf{n}(t)\}$, we can group terms two by two and write the sum as:

$$\begin{aligned} &\sum_i \mathcal{A}\{\mathbf{n}_{2i}(t)\} + \mathcal{A}\{\mathbf{n}_{2i-1}(t)\} = \\ &\sum_i \mathcal{A}\{\sqrt{1 - a^2 \mathbf{l}_{2i}^2}\mathbf{m}_{2i}(t) + a\mathbf{l}_{2i}(t)\} \\ &-\mathcal{A}\{\sqrt{1 - a^2 \mathbf{l}_{2i-1}^2}\mathbf{m}_{2i-1}(t) - a\mathbf{l}_{2i-1}(t)\}. \end{aligned}$$

We can now use the expression relating the difference in the areas produced by curves $\mathbf{n}(t)$ and $\mathbf{n}(t)+\delta\mathbf{n}(t)$:

$$\mathcal{A}\{\mathbf{n}(t)+\delta\mathbf{n}(t)\} = \mathcal{A}\{\mathbf{n}(t)\} + \int_0^\beta dt\ \delta\mathbf{n}(t).\left(\partial_t(\mathbf{n}(t)) \times \mathbf{n}(t)\right)$$

and we obtain

$$\sum_i \mathcal{A}\{\mathbf{n}_{2i}(t)\} + \mathcal{A}\{\mathbf{n}_{2i-1}(t)\} =$$

$$-a\int_0^\beta dt\ \sum_i \left(\frac{\Delta}{a}(\mathbf{m}_{2i}(t)) + 2\mathbf{l}_{2i}(t)\right).\left(\mathbf{m}_{2i}(t)\times\partial_t(\mathbf{m}_{2i}(t))\right) + O(a^2) \quad (6.12)$$

where we have used that $\mathbf{m}_{2i-1}(t) = \mathbf{m}_{2i}(t) - \Delta(\mathbf{m}_{2i}(t)) + O(a^2)$.

In the same spirit, and omitting constant terms, the second term in (6.10) can be written as:

$$\frac{JS^2}{2}\int_0^\beta dt \sum_i \left[\left(\mathbf{n}_{2i}(t)+\mathbf{n}_{2i+1}(t)\right)^2 + \left(\mathbf{n}_{2i+1}(t)+\mathbf{n}_{2i+2}(t)\right)^2\right]$$

and the lowest order in a gives:

$$\frac{JS^2a^2}{2}\int_0^\beta dt \sum_i \left\{\left[-\frac{\Delta}{a}\mathbf{m}_{2i}(t)+2\mathbf{l}_{2i}\right]^2 + \left[\frac{\Delta}{a}\mathbf{m}_{2i+1}(t)+2\mathbf{l}_{2i+1}\right]^2\right\},$$

which, still to lowest order in a can also be written as:

$$JS^2a^2\int_0^\beta dt \sum_i \left[(\frac{\Delta}{a}\mathbf{m}_{2i}(t))^2 + 4\mathbf{l}_{2i}^2\right].$$

We can now collect all the pieces together and take the continuum limit by replacing $\frac{\Delta}{a}\to\partial_x$, $2a\sum_i \to \int dx$ (the factor of 2 arises from the doubling of the chain index). We also take the limit of zero temperature $T\to 0$. We obtain for the total action:

$$S[\mathbf{m},\mathbf{l}] = \frac{JS^2a}{2}\int dx\ dt\ \left[(\partial_x(\mathbf{m}(x,t))^2 + 4\mathbf{l}(x,t)^2\right]$$

$$+\frac{iS}{2}\int dxdt\,\left(\partial_x(\mathbf{m}(x,t)) + 2\mathbf{l}(x,t)\right)\cdot\left(\mathbf{m}(x,t)\times\partial_t(\mathbf{m}(x,t))\right). \quad (6.13)$$

We immediately notice that this action is quadratic in the variable $\mathbf{l}$. We can then integrate out this variable and obtain the final result:

$$S[\mathbf{m}] = \int dxdt\frac{1}{2g}\left(v(\partial_x\mathbf{m})^2 + \right.$$

$$\left.\frac{1}{v}(\partial_t\mathbf{m})^2 + \frac{i\theta}{8\pi}\epsilon_{ij}\mathbf{m}\cdot(\partial_i\mathbf{m}\times\partial_j\mathbf{m})\right) \quad (6.14)$$

with $g = 2/S$ the coupling constant, $v = 2aJS$ the spin wave velocity and the topological angle $\theta = 2\pi S$. If we want the action to be finite in an infinite system and at zero temperature, we have to impose that $\mathbf{m}$ tends to a fixed vector $\mathbf{m}_0$ at infinity in space and imaginary time. By making all the points at infinity equivalent, we are just saying that our space time is equivalent to a sphere S_2. Since in each point of the space time $\mathbf{m}$ can also be viewed as an element of S_2, the mapping $\mathbf{m}(x,t)$ corresponds to an embedding of the sphere into itself. Such embeddings are classified by what is called the second homotopy group of the sphere $\Pi_2(S_2) = \mathbb{Z}$ [3]. To each embedding corresponds an integer (element of $\mathbb{Z}$) given by the Pontryagin index:

$$\frac{1}{8\pi}\int dxdt\ \epsilon_{ij}\mathbf{m}\cdot(\partial_i\mathbf{m}\times\partial_j\mathbf{m}) \in \mathbb{Z} \tag{6.15}$$

where we immediately recognize in this expression the last term of the action (6.14). We can then conclude from this result that for integer S, the imaginary part of the action in (6.14) (which we will call the topological term) is always a multiple of 2π and plays no role at all, while for half integer spins, as we will see, the situation is completely different.

6.4 The Hamiltonian Approach

The result (6.14) can also be derived using a Hamiltonian approach; we follow here the derivation given in [4], [5]. Let us group our spin operators two by two and define the variables $\mathbf{L}$ and $\mathbf{M}$ through

$$\begin{aligned}\mathbf{S}_{2i} &= a\mathbf{L}_i - S\mathbf{M}_i\\ \mathbf{S}_{2i+1} &= a\mathbf{L}_i + S\mathbf{M}_i.\end{aligned} \tag{6.16}$$

These relations can be inverted:

$$\begin{aligned}\mathbf{L}_i &= \frac{1}{2a}\left[\mathbf{S}_{2i+1}+\mathbf{S}_{2i}\right]\\ \mathbf{M}_i &= \frac{1}{2S}\left[\mathbf{S}_{2i+1}-\mathbf{S}_{2i}\right]\end{aligned} \tag{6.17}$$

and one can then easily show using (6.1) that these variables satisfy the constraints:

$$a^2\mathbf{L}_i^2 + S^2\mathbf{M}_i^2 = S(S+1)\ ;\ \mathbf{L}_i\cdot\mathbf{M}_i = 0 \tag{6.18}$$

and the algebra:

$$\begin{aligned}[L_i^a, L_j^b] &= \frac{i}{2a}\epsilon^{abc}\delta_{i,j}L_i^c\ ;\\ [L_i^a, L_j^b] &= \frac{i}{2a}\epsilon^{abc}\delta_{i,j}M_i^c\ ;\end{aligned}$$

$$[L_i^a, L_j^b] == \frac{ia}{2S^2}\epsilon^{abc}\delta_{i,j}L_i^c. \tag{6.19}$$

We can rewrite the Hamiltonian (6.9) as:

$$\begin{aligned} H &= J\sum_i [\mathbf{S}_{2i}\cdot\mathbf{S}_{2i+1} + \mathbf{S}_{2i+1}\cdot\mathbf{S}_{2i+2}] \\ &= J\sum_i \big[-S^2\mathbf{M}_i^2 + a^2\mathbf{L}_i^2 + a^2\mathbf{L}_i\cdot\mathbf{L}_{i+1} + \\ aS\,(\mathbf{M}_{i-1}\cdot\mathbf{L}_i &- \mathbf{L}_i\cdot\mathbf{M}_{i+1}) + \frac{S^2}{2}(\mathbf{M}_i - \mathbf{M}_{i+1})^2 - S^2\mathbf{M}_i^2\Big] \end{aligned} \tag{6.20}$$

where the index of the first term of the last line has been shifted by one for convenience. To make contact with the result of the preceding section, we take the continuum limit by keeping in the Hamiltonian only the terms of order a^2. We start by defining the variable x as:

$$\mathbf{M}_i \to \mathbf{M}(x)\ ;\ \mathbf{M}_{i\pm1} \to \mathbf{M}(x) \pm 2a\partial_x(\mathbf{M}(x)) + O(a^2).$$

Using then the identification

$$2a\sum_i \to \int dx\ ;\ \frac{1}{2a}\delta_{i,j} \to \delta(x-y)$$

and the relation (6.18), we obtain the continuous Hamiltonian (we omit constant terms):

$$H = \frac{v}{2}\int dx\left[g\left(\mathbf{L} - \frac{\theta}{4\pi}\partial_x(\mathbf{M})\right)^2 + \frac{1}{g}\left(\partial_x(\mathbf{M})\right)^2\right] \tag{6.21}$$

where g, v and θ have already been defined. The key point is to realize that for $S \to \infty$, the constraint and the algebra become:

$$\mathbf{M}^2(x) = 1\ ;\ \mathbf{L}(x)\cdot\mathbf{M}(x) = 0 \tag{6.22}$$

$$[L^a(x), L^b(y)] = i\epsilon^{abc}\delta(x-y)L^c(x)\ ;\ [L^a(x), M^b(y)] = i\epsilon^{abc}\delta(x-y)M^c(x)$$

$$[M^a(x), M^b(y)] = 0 \tag{6.23}$$

and in this limit we can view $\mathbf{L}(x)$ as the angular momentum density $\mathbf{M}(x)\times\dot{\mathbf{M}}(x)$ associated to the normalized field $\mathbf{M}$ (note the similarity between this operator relation and the ansatz (6.11)) . Upon the replacement $\mathbf{m} \to \mathbf{M}$ in (6.14), and using an appropriate parametrization for this normalized field (as, for example, the azimuthal angles in the sphere), one can easily show that (6.21) is the Hamiltonian associated to the Lagrangian of the action (6.14), which completes our alternative derivation of the non-linear sigma model description of Heisenberg antiferromagnetic chains in the large S limit.

6.5 The Non-linear Sigma Model and Haldane's Conjecture

Let us first consider the case of integer spins, where the topological term is absent. We then have the usual $O(3)$ non linear sigma model (NLSM)[4] for which we can write the partition function as:

$$Z = \int D\{\mathbf{m}\}\delta(\mathbf{m}^2 - 1)e^{-\int dxdt \frac{1}{2g}\left((\partial_x \mathbf{m})^2 + (\partial_t \mathbf{m})^2\right)} \tag{6.24}$$

and where we have set the sound velocity v to unity. At the classical level, the action in (6.24) is scale invariant. Since there is no apparent scale parameter in the model, one would be tempted to conclude that the correlation function of this model are algebraically decaying, a phenomenon typical of scale invariant systems. We will see however that fluctuations change dramatically this scenario [6]. To see this, we start by expressing the δ function in (6.24) in terms of a Lagrange multiplier:

$$Z = \int_{c-i\infty}^{c+i\infty} D\{\lambda(x)\} \int D\{\mathbf{m}(x)\}e^{-\int dxdt \frac{1}{2g}\left((\partial_x \mathbf{m})^2 + (\partial_t \mathbf{m})^2 + \lambda(\mathbf{m}^2 - 1)\right)} . \tag{6.25}$$

In order to understand the qualitative behavior of (6.24), we are going to do an approximation which consists in replacing the integral in λ by the maximal value of the integrand:

$$Z \sim \int D\{\mathbf{m}(x)\}e^{-\int dxdt \frac{1}{2g}\left((\partial_x \mathbf{m})^2 + (\partial_t \mathbf{m})^2 + \lambda_m(\mathbf{m}^2 - 1)\right)} , \tag{6.26}$$

where the optimal value λ_m is assumed to be a constant. As we will see below, such an approximation is valid if we generalize our model to the $O(N)$ non-linear sigma model and consider the limit $N \gg 1$. The approximate partition function in (6.26) has the advantage of being Gaussian and then all physical quantities can be easily calculated.

To obtain λ_m, we integrate over $\mathbf{m}(x)$ in (6.25):

$$Z = \int_{c-i\infty}^{c+i\infty} D\{\lambda(x)\}e^{\frac{1}{2g}\left(\int \lambda(x,t)dxdt - \frac{N}{2}\log(\det\{-\triangle + \lambda(x,t)\})\right)} , \tag{6.27}$$

where $\triangle$ is the two dimensional Laplace operator. It is now apparent that for $N \gg 1$ we can estimate this integral by a saddle-point approximation. The condition for maximizing the integrand is:

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[4] The historical origin of this name comes from the way the field was written in some choice of variables where the $O(N)$ symmetry is realized non-linearly.

$$\frac{1}{2g} = \frac{N}{2}\frac{\delta}{\delta\lambda(x,t)}\log(\det\{-\triangle + \lambda(x,t)\}) \tag{6.28}$$

And, again, under the assumption of λ_m being constant, we get:

$$\begin{aligned} 1 &= gNTr\{\frac{1}{-\triangle + \lambda_m}\} \\ &= gN\int \frac{d^2p}{4\pi^2}\frac{1}{p^2+\lambda_m} \\ &= \frac{gN}{4\pi} log(\Lambda^2/\lambda_m) \end{aligned} \tag{6.29}$$

where Λ is an ultraviolet momentum cut-off. From (6.29) we obtain the optimal value:

$$\lambda_m = \Lambda^2 e^{-\frac{4\pi}{gN}} \tag{6.30}$$

which indicates us that fluctuations have dynamically generated a mass term in our original action. Indeed, by using (6.26) one easily sees that correlation functions are now exponentially decaying with the distance.

The arguments we used to derive the result (6.30) are strictly speaking valid for $N \gg 1$. It is however well established by many techniques that this result is indeed qualitatively correct for $N \geq 3$. The non-linear sigma model is integrable even at the quantum level and an exact S matrix has being proposed [7]. An intuitive way to understand this result is by seeing (6.24) as the partition function of a classical magnet in the continuum. In such an interpretation, the coupling g plays the role of temperature. Another approach to understand this phenomenon is given by the renormalization group analysis. We refer the reader to [6], [8] for a detailed presentation of the renormalization group techniques and give here the main steps of the procedure. The idea is to decompose the field in slowly and fast fluctuating parts; we then integrate over the fast degrees of freedom to obtain an effective action with renormalized parameters. Following [6] we start by writing our field as:

$$\mathbf{m} = \sqrt{1 - \sum_i \sigma_i^2}\mathbf{m}_s + \sum_{i=1}^{N-1} \sigma_i \mathbf{e}_i, \tag{6.31}$$

where $\mathbf{m}_s^2 = 1$ and the vectors $\mathbf{e}_i$ form an orthonormal basis for the space orthogonal to $\mathbf{m}_s$, and the fields σ_i are the fast fluctuating degrees of freedom. Keeping only the quadratic terms in the fields σ_i, the action (6.14) becomes:

$$\int dxdt \frac{1}{2g}\left[\sum_\mu \sum_{i,j} (\partial_\mu \sigma_i - \partial_\mu(\mathbf{e}_i)\cdot \mathbf{e}_j \sigma_j)^2 \right.$$

$$\left. \sum_\mu \sum_{i,j} (\partial_\mu \mathbf{m}_s).\mathbf{e}_i (\partial_\mu \mathbf{m}_s).\mathbf{e}_j (\sigma_i\sigma_j - \sum_k \sigma_k^2 \delta_{ij}) + \sum_\mu (\partial_\mu \mathbf{m}_s)^2 \right] . \tag{6.32}$$

We can now integrate over the fast degrees of freedom in the momentum shell $\Lambda - \delta\Lambda < p < \Lambda$, or in real space between scales L and $L + \delta L$ to obtain an effective coupling for the slow part of the action $\sum_\mu (\partial_\mu \mathbf{m}_s)^2$. To lowest order in g, the contribution to each component $(\partial_\mu \mathbf{m}_s^i)^2$, of the slow part of the action is given by the one point function:

$$\langle \sigma_i \sigma_i - (N-1) \sum_k \sigma_k^2 \rangle$$

and after some algebra, we can do the integration to obtain the new coupling:

$$\frac{1}{g + \delta g} = \frac{1}{g} - (N-2) g^2 \int_{\Lambda - \delta\Lambda}^{\Lambda} \frac{d^2 p}{(2\pi)^2 p^2} . \tag{6.33}$$

An important observation is that the term $\partial_\mu(\mathbf{e}_i) \cdot \mathbf{e}_j$ does not contribute to this result. The way to understand this is to notice that the action is invariant under rotations in the $N-1$ dimensional space $\alpha_i \to M_{ij} \alpha_j$ with $\alpha_i = \sigma_i$, $(\partial_\mu \mathbf{m}_s) \cdot \mathbf{e}_i$ and the term $\partial_\mu(\mathbf{e}_i) \cdot \mathbf{e}_j$ behaves under this transformation as a gauge field. Since $\sum_\mu (\partial_\mu \mathbf{m}_s)^2$ is rotationally invariant, the lowest order contribution we can have in the effective action arising from $\partial_\mu(\mathbf{e}_i) \cdot \mathbf{e}_j$ is the (gauge) invariant term $\sum_{\mu,\nu,i,j} (\partial_\mu(\mathbf{e}_i) \cdot \partial_\nu(\mathbf{e}_j) - \partial_\mu(\mathbf{e}_i) \cdot \partial_\nu(\mathbf{e}_j))^2$. This gauge invariant term give rise to non-logarithmic divergences which can be shown to give no contribution in (6.33). From this result we obtain the β function:

$$\beta(g) = -\frac{dg}{dln(\Lambda)} = \frac{dg}{dln(L)} = \frac{N-2}{2\pi} g^2 + O(g^3). \tag{6.34}$$

That is, by going to large scales g flows to strong coupling indicating a regime of high temperature where the system is disordered and with a finite correlation length. Of course the original cut-off Λ and coupling g depend on the microscopic details of the theory, but if we imagine varying such parameters in our field theory in such a way to keep the dynamically generated scale constant, the constant λ satisfies the equation:

$$\frac{\partial \lambda}{\partial \Lambda} + \frac{\partial \lambda}{\partial g} \frac{\partial g}{\partial \Lambda} = 0 . \tag{6.35}$$

Using (6.34) and assuming that $\lambda = \Lambda^2 f(g)$, we obtain:

$$\lambda = \Lambda^2 e^{-\frac{4\pi}{g(N-2)}} \tag{6.36}$$

which coincides with (6.30) for $N \to \infty$. Remember that our case of interest corresponds to $N = 3$.

The case of half-integer spins is very different. We have to remember that in this case the action (6.14) contain the topological term which contributes as a destructive phase term in the computation of the partition function. The coupling constant still flows to the strong coupling regime, but the topological term is protected against renormalization because of its discrete nature. Recalling that the coupling constant $g = 2/S$ is inversely proportional to the spin S, the flow to strong coupling can be interpreted as a large scale behavior of the system with smaller spins. Since the topological term remains present at large scale, one can conclude that the large scales behavior of half integer spin chains corresponds to the one of spin 1/2 chain, which is known to be gapless. Shankar and Read have given further support to this conclusion [9]. This drastic difference between integer and half integer antiferromagnetic spin chains is known as the Haldane conjecture [10].

6.6 Antiferromagnetic Spin Ladders

The techniques we have used so far to obtain the large scales behavior of Heisenberg antiferromagnetic chains can be generalized to other geometries. The closest example is given by the spin ladder systems. Imagine an array of spins forming a strip composed of N chains. Neighboring spins belonging to the same chains are supposed to have a coupling J, as before, while neighboring spins of adjacent chains have a coupling given by J'. We assume also that both couplings J and J' are positive. The spins on this ladder are labelled by the chain index n and the row index a, $1 \leq a \leq N$. N is to be considered as fixed and finite while the number of spins along the chains diverge in the thermodynamic limit. The analysis presented in this section follows the lines of [5].

At the classical level, the lowest energy configurations are given by a Néel order, say, in the $\hat{z}$ direction given by:

$$\mathbf{S}_{a,n} = (-1)^{a+n} S\hat{z}. \tag{6.37}$$

The equations of motion for a spin belonging to an intermediate row is given by:

$$\frac{d\mathbf{S}_{a,n}}{dt} = -\mathbf{S}_{a,n} \times \left[J\left(\mathbf{S}_{a,n-1} + \mathbf{S}_{a,n+1}\right) + J'\left(\mathbf{S}_{a-1,n} + \mathbf{S}_{a+1,n}\right)\right], \tag{6.38}$$

(for the spins belonging to the edges rows, the $a-1$ or $a+1$ terms are absent). We can now use our experience in spin wave analysis to identify the low energy excitations around this Néel state. We linearize (6.38) and write the ansatz:

$$S^x_{a,n} + iS^y_{a,n} = e^{i(wt+nq)}\left(A_a(q) + (-1)^{a+n+1} B_a(q)\right). \tag{6.39}$$

Note that the example of decoupled chains is a particular case of this model. With the ansatz proposed here one must of course recover the well-known results for the simple chain in the limit $J' \to 0$. The resulting eigenvalue problem shows that there are $N-1$ families of solutions, which for $J' \neq 0$ have a gapped spectrum (one can show however that such modes become gapless for the case of decoupled chains $J' = 0$). There is only one family of excitations with vanishing energy at $q = 0$ and $q = \pi$. These modes are the counterpart of the gapless modes of the single Heisenberg chain and give us a clue of the form of the slowly varying fields in a field theory approach.

The solution for small q is given by $B_a = B$ and $A_a \propto q \sum_b L^{-1}_{ab}$ with

$$L = \begin{pmatrix} 4J + J' & J' & 0 & .. \\ J' & 4J + 2J' & J' & .. \\ 0 & J' & 4J + 2J' & .. \\ .. & .. & .. & .. \end{pmatrix} . \tag{6.40}$$

We can now work out the path integral description of the low energy physics of the ladder system. We refer the reader to [5] for the Hamiltonian derivation of it and the subtle differences between the path integral and Hamiltonian results. By using our basis of states $|\mathbf{n}\rangle$ in (6.2), we write the action of the ladder as:

$$S[\{\mathbf{n}_{a,n}\}] = -iS \sum_{a,n} \mathcal{A}\{\mathbf{n}_{a,n}(t)\} + JS^2 \int_0^\beta dt \sum_{n,a} \mathbf{n}_{a,n}(t) \cdot \mathbf{n}_{a,n+1}(t)$$
$$+ J'S^2 \int_0^\beta dt \sum_n \sum_{a=1}^{N-1} \mathbf{n}_{a,n}(t) \cdot \mathbf{n}_{a+1,n}(t) \tag{6.41}$$

and, inspired by the spin wave result, we propose as an ansatz the generalization of (6.11):

$$\mathbf{n}_{a,n} = (-1)^n \sqrt{1 - a^2\alpha_a^2 \mathbf{l}(n)^2}\ \mathbf{m} + a\alpha_a \mathbf{l}(n) \tag{6.42}$$

where $\alpha_a = A_a / (\sum_a A_a)$. Note that excitations along the transverse direction of the ladder are all supposed to be of high energy. This is due to the fact that N is kept finite implying a finite difference in the energy levels of transverse excitations. Then, the effective low energy degrees of freedom are one-dimensional in nature. The procedure is then standard: we insert expression (6.42) into (6.41) and work out the continuum limit. The final result is again the effective action (6.14) with the parameters:

$$g = \frac{1}{S\sqrt{\sum_{a,b} L^{-1}_{ab}}}\ ;\ v = \frac{SJa}{\sqrt{\sum_{a,b} L^{-1}_{ab}}}\ ;\ \theta = 2\pi S \sum_{a=1}^{N} (-1)^a . \tag{6.43}$$

The important result here is the contribution to the topological term which is easy to understand by noticing that, with the assumption we made in

(6.42), each chain will contribute to it with a term $2\pi S(-1)^a$. We then see that only for an odd number of coupled chains, and for half-integer spin will the topological term give rise to a gapless behavior of our system at zero temperature. To summarize, the Non-Linear Sigma model approach predicts that the large scale behavior of the system is governed by the product SN: if it is an integer, the system is expected to be gaped, while for half-integer values, the system is gapless.

6.7 Chains with Alternating Bonds

As another example of the applications of the NLSM technique, we can consider the study of spin chains with alternating couplings, or dimerization [4] [5] with the Hamiltonian:

$$J\sum_i \left[(1+\delta)\mathbf{S}_{2i}(t)\cdot\mathbf{S}_{2i+1}(t) + (1-\delta)\mathbf{S}_{2i+1}(t)\cdot\mathbf{S}_{2i+2}(t)\right]. \qquad (6.44)$$

We can use both the path integral or Hamiltonian approach to obtain the effective action in the continuum limit. Within this last approach, we use again the operators (6.17). It is a straightforward computation to show that the Hamiltonian is now:

$$\begin{aligned} H = J\sum_i \Big[& -(1+\delta)S^2\mathbf{M}_i^2 + a^2(1+\delta)\mathbf{L}_i^2 + a^2(1-\delta)\mathbf{L}_i\cdot\mathbf{L}_{i+1} + \\ & aS(1-\delta)\left(\mathbf{M}_{i-1}\cdot\mathbf{L}_i - \mathbf{L}_i\cdot\mathbf{M}_{i+1}\right) + \\ & \frac{S^2}{2}(1-\delta)(\mathbf{M}_i - \mathbf{M}_{i+1})^2 - S^2(1-\delta)\mathbf{M}_i^2 \Big] \end{aligned} \qquad (6.45)$$

and completing squares and taking the continuum limit as before, we obtain (see [4]):

$$H = \frac{\tilde{v}}{2}\int dx \left[\tilde{g}\left(\mathbf{L} - \frac{\tilde{\theta}}{4\pi}\partial_x(\mathbf{M})\right)^2 + \frac{1}{\tilde{g}}\left(\partial_x(\mathbf{M})\right)^2\right] \qquad (6.46)$$

with now $\tilde{g} = 2/(S\sqrt{1-\delta^2})$, $\tilde{v} = 2aJS\sqrt{1-\delta^2}$ and $\tilde{\theta} = 2\pi S(1-\delta)$. Building the corresponding Lagrangian we observe that the resulting sigma model has now a topological term with a factor of $1-\delta$ in front. This result can be easily obtained also within the path integral approach. The topological term changes sign under a parity transformation, as well as time reversal and $\mathbf{m} \to -\mathbf{m}$. In the non–dimerized case (which is parity invariant) this fact has no importance since the factor in front of it is a multiple of π and an overall sign has no effect in the computation of the partition function. The situation is different in the presence of dimerization. Now the total action is not anymore invariant under such transformation.

One important question is what is the large scales behavior of the NLSM in the presence of a topological term with coefficient different from $\pm\pi$. We refer the reader to [4] to a discussion about this delicate issue and just announce the commonly believed scenario: the NLSM is massless only for $\theta = \pm\pi$. A result in support of the idea that for a nontrivial θ we obtain a massive behavior is the fact that a spin 1/2 chain with dimerization has a gap in the spectrum, as we are going to see below in the context of bosonization. Based in this belief we can then conclude that by varying the parameter δ one should encounter $2S + 1$ gapless points in a spin S chain. Such results can also be extended to the case of spin ladders where different kinds of dimerizations are conceivable [5], and where, again, one recover a NLSM with a non-integer factor for the topological term.

6.8 The Two-Dimensional Heisenberg Antiferromagnet

We start our discussion on two-dimensional antiferromagnets by considering spins S located at the vertices of a square lattice $\mathbf{S}_{i,j}$, where i and j label the position on the lattice for each spin. The Hamiltonian is:

$$H = J \sum_{i,j} \mathbf{S}_{i,j} \cdot (\mathbf{S}_{i+1,j} + \mathbf{S}_{i,j+1}). \tag{6.47}$$

We are going to consider again the $T \to 0$ limit. Within the path integral approach, the effective action is given by:

$$S[\{\mathbf{n}_{i,j}\}] = -iS \sum_{i,j} \mathcal{A}\{\mathbf{n}_{i,j}(t)\} + JS^2 \int dt \sum_{i,j} \mathbf{n}_{i,j}(t) \cdot (\mathbf{n}_{i+1,j}(t) + \mathbf{n}_{i,j+1}(t)). \tag{6.48}$$

We are going again to make use of the result of spin wave theory and assume that, for large S, the low-energy physics of the system can be described by the ansatz:

$$\mathbf{n}_{i,j} = (-1)^{i+j} \sqrt{1 - a^2 \mathbf{l}_{i,j}^2}\; \mathbf{m}_{i,j} + a\mathbf{l}_{i,j}. \tag{6.49}$$

Before obtaining explicitly the effective action arising from this ansatz, let us discuss first which kind of topological terms one can expect in the computation of the partition function. In order to have a finite value for the action, we assume again that the field configuration tends to the same constant field at spatial and imaginary time infinity. By associating all the points at infinity, the space-(imaginary)time manifold corresponds now to S^3. On the other hand, the order parameter field is still en element of S^2. The possibility of having configurations of the spin field with non-trivial winding is given by the homotopy group $\Pi_2(S^3) = 1$ which turns out to be trivial. One can then already see that the specifics in the physics of one-dimensional systems are not recovered in the square lattice. The situation can be more subtle for frustrating systems, like the Heisenberg antiferromagnet in the triangular lattice. In this case the classical Néel configuration is obtained by imposing that adjacent spins in each triangle form a planar configuration with a relative angle of $2\pi/3$. The orientation of such triad is characterized by an element of $SO(3)$: we have to specify a vector orthogonal to the plane of the triad and the angle that forms on this plane the triad with respect to a reference configuration. Since $\Pi_2(SO(3)) = \mathbb{Z}$, we can expect in this case to have non-trivial contributions to the partition function from a topological origin. A microscopic derivation of the effective action has revealed indeed the possibility of such kind of non-trivial contributions [11], but its consequences in the large scale physics are much less easy to understand than in the one-dimensional case.

Let us now resume our discussion about the antiferromagnet in the square lattice, where topology can still play a rôle. Any field configuration at a given time can be characterized by an integer corresponding to the Pontryagin index that we have discussed before:

$$\frac{1}{4\pi}\int dxdy\mathbf{m}\cdot(\partial_x\mathbf{m}\times\partial_y\mathbf{m}). \tag{6.50}$$

If the field $\mathbf{m}(x,y,t)$ varies smoothly with the time, this quantity keeps the same integer value all along the time. This quantity corresponds to the total charge of textural defects of the field configuration, called skyrmions [12]. The presence of such a term in the effective action would have dramatic consequences on the statistic of such skyrmions. Haldane [13] has shown however that the effective action of the square lattice antiferromagnet has no such topological terms. He considered however the possibility of singular configurations of the field allowing for tunneling processes that change this integer index. Such a kind of singularity, called a hedgehog, can play a rôle if the system is disordered and Haldane found a non-trivial S dependence of that term on the basis of a microscopic derivation of the effective action. We limit ourselves in this discussion to the case of non-singular configurations of the field and derive the effective action arising from the ansatz (6.49).

The part arising from the Hamiltonian can be treated in the same spirit as in the one-dimensional case and we have:

$$JS^2\int dt\sum_{i,j}\mathbf{n}_{i,j}(t)\cdot(\mathbf{n}_{i+1,j}(t)+\mathbf{n}_{i,j+1}(t)) =$$

$$\frac{JS^2}{2}\int dt\left[\sum_{i,j}(\mathbf{n}_{i,j}(t)+\mathbf{n}_{i+1,j}(t))^2+\sum_{i,j}(\mathbf{n}_{i,j}(t)+\mathbf{n}_{i,j+1}(t))^2\right]$$

which in the continuum limit gives:

$$\frac{JS^2}{2}\int dt\int dx\int dy\left((\partial_x\mathbf{m})^2+(\partial_y\mathbf{m})^2+8\mathbf{l}^2\right).$$

The outcome of the area term is a bit more subtle. We start by grouping the contribution of spins two by two along, say, the $\hat{x}$ direction, as we did in the one-dimensional case and we get:

$$-a\int dt\ \sum_{i,j}\left((-1)^j\frac{\Delta_i}{a}(\mathbf{m}_{2i,j}(t))+2\mathbf{l}_{2i,j}(t)\right)\cdot(\mathbf{m}_{2i,j}(t)\times\partial_t(\mathbf{m}_{2i,j}(t)))+O(a^2) \tag{6.51}$$

where Δ_i stands for the difference (or lattice derivative) in the i ($\hat{x}$) direction. We know that the term

$$\int dt\sum_i\left(\frac{\Delta_i}{a}(\mathbf{m}_{2i,j}(t))\right)\cdot(\mathbf{m}_{2i,j}(t)\times\partial_t(\mathbf{m}_{2i,j}(t)))$$

is going to give rise to the integer associated to the Pontryagin index in the x-t space-time slice. Since the field $\mathbf{m}$ is assumed to be slowly varying and non-singular, this integer must be the same for each row j. Then, because of the alternating sign in the sum in (6.51), this term cancels. In the continuous limit, the only contribution from the area term is then:

$$\frac{iS}{a}\int dt\int dx\int dy\mathbf{l}(x,y,t)\cdot(\mathbf{m}(x,y,t)\times\partial_t(\mathbf{m}(x,y,t)))\,.$$

Collecting all the terms together and, again, integrating over the field $\mathbf{l}$ we obtain the final result for the action:

$$S=\frac{1}{2g}\int dxdydt\left(v\left[(\partial_x\mathbf{m})^2+(\partial_y\mathbf{m})^2\right]+\frac{1}{v}(\partial_t\mathbf{m})^2\right) \tag{6.52}$$

with $g=2\sqrt{2}a/S$, $v=2\sqrt{2}aJS$. To understand the behavior of this action, we start by noticing that the partition function is equivalent to that of a continuous magnet in three dimensions. Again g plays the rôle of a temperature and one expects the existence of some critical value below which the $O(3)$ symmetry is broken.

To see this in more detail, we proceed as in the (1+1) dimensional case and consider the $O(N)$ non-linear sigma model. The procedure is strictly the same, and we obtain again the saddle-point equation:

$$\begin{aligned}1&=gN\mathrm{tr}\{\frac{1}{-\triangle+\lambda_m}\}\\&=gN\int\frac{d^3p}{(2\pi)^3}\frac{1}{p^2+\lambda_m}\end{aligned} \tag{6.53}$$

where now the integral over momenta is three-dimensional. As in the (1+1) case, this integral is divergent at high momenta and has to be regularized by a cut-off $\Lambda\sim\frac{1}{a}$. By a careful inspection of the integral (6.53), one can see that there is a real and strictly positive solution for λ_m for any value of g bigger than the critical value g_c obtained from:

$$1 = g_c N \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2} = \frac{g_c N \Lambda}{2\pi^2}. \tag{6.54}$$

For any $g > g_c$ the scenario is similar to the (1+1) dimensional case where the symmetry is unbroken and excitations acquire a gap.

If $g < g_c$, there is no real and positive solution of (6.53). This phase corresponds to the broken symmetry phase we mentioned above. This scenario is also supported by a one loop computation of the β function in $2+\epsilon$ dimensions (and setting $\epsilon = 1$ here):

$$\beta(g) = -g + \frac{N-2}{2\pi} g^2 + ...$$

(recall that now g is a dimension-full constant). This result suggests that there is a critical value of g which is the only point in which the system is truly scale invariant. Below that value the system flows to the low temperature phase and above it it flows to the high temperature phase.

For very small g, and taking back $N = 3$ we can describe our field as a small deformation of an homogeneous vector, say, in the $\hat{z}$ direction:

$$\mathbf{m}(x,y,t) = (\sqrt{1-\alpha_1^2-\alpha_2^2}, \alpha_1(x,y,t), \alpha_2(x,y,t))$$

and the remaining action

$$S \sim \frac{1}{2g} \sum_{a=1}^{2} \int dxdydt \left(c\left[(\partial_x \alpha_a)^2 + (\partial_y \alpha_a)^2 \right] + \frac{1}{c} (\partial_t \alpha_a)^2 + ... \right) \tag{6.55}$$

is simply the one of two massless Goldstone modes. We thus conclude that there must be a critical value of the coupling constant, proportional to a that separates the ordered from the disordered phase. This means that there must be a critical value of the spin magnitude S_c above which the system is ordered at zero temperature. Since we have by now numerical and experimental evidence that the spin 1/2 Heisenberg antiferromagnet has an ordered ground state, we then conclude that it is ordered for all values of S at $T = 0$.

6.9 Bosonization of 1D Systems

6.9.1 *XXZ* Chain in a Magnetic Field: Bosonization and Luttinger Liquid Description

We consider now a generalization of the one-dimensional $SU(2)$ Hamiltonian (6.9) for $S = 1/2$, by including an anisotropy term in the z direction, which we parameterize by Δ, and an external magnetic field h applied along the z-axis. The resulting model is known as the XXZ chain which, being integrable, allows for a detailed analysis of the low energy theory using abelian bosonization. This simple theory also serves as a starting point for the study of many different situations which can be described by its perturbations as the case of modulated chains or N leg ladders made up of XXZ chains. Given its importance, we present the bosonization analysis in detail.

The lattice Hamiltonian is given by

$$H_{XXZ}^{latt} = J \sum_n \left(\frac{1}{2} \left(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+ \right) + \Delta S_n^z S_{n+1}^z \right) - h \sum_n S_n^z . \quad (6.56)$$

where we consider $J > 0$. $S_n^\pm = S_n^x \pm i S_n^y$ are the spin raising and lowering operators where $S_n^{x,y,z}$ are the spin operators acting on site n and satyisfying the $SU(2)$ algebra (6.1). In this section we restrict ourselves to $S = 1/2$.

This model has a $U(1)$ invariance corresponding to rotations around the internal z axis for generic Δ. For $\Delta = 0$ we have the XY model which can be solved exactly using the Jordan-Wigner transformation and it is the starting point of the bosonization procedure that we describe below. The full $SU(2)$ spin symmetry is recovered at $\Delta = 1$ and $h = 0$ where it is more convenient to apply non-abelian bosonization. This case will be discussed in Sect. 9.3.

We first summarize the outcome of the bosonization of the XXZ chain and then present its derivation in detail. For a complete bibliography see [4, 14–17] and references therein.

The Hamiltonian (6.56) is exactly solvable by Bethe ansatz and it can be shown that its low-energy properties are described by a scalar boson with a Hamiltonian given by

$$H_{XXZ}^{cont} = \frac{v}{2} \int \mathrm{d}x \left(K \left(\partial_x \tilde{\phi}(x) \right)^2 + \frac{1}{K} \left(\partial_x \phi(x) \right)^2 \right) \quad (6.57)$$

where $\tilde{\phi}$ is the field dual to the scalar field ϕ and it is defined in terms of its canonical momentum as $\partial_x \tilde{\phi} = \Pi$. This notation is usually introduced in order to simplify the expressions of the spin operators in the continuum limit; see the Appendix for details on our conventions.

The Fermi velocity v and the so-called Luttinger parameter K depend on both the magnetic field and the anisotropy parameter Δ. These two parameters determine completely the low energy dynamics of the lattice model and they can be computed from the Bethe Ansatz solution. For zero magnetic field and $-1 < \Delta < 1$ they can be found in closed form:

$$K(\Delta) = \frac{\pi}{2(\pi - \theta)} \qquad v(\Delta) = \frac{\pi}{2} \frac{\sin\theta}{\theta} \quad (6.58)$$

where $\cos(\theta) = \Delta$ and we have set $J = 1$. Otherwise, one has to solve numerically a set of integro-differential equations (see [18]) which result is discussed below.

The Hamiltonian (6.57) corresponds to a conformal field theory with central charge $c = 1$ and the free boson is compactified at radius R, *i.e.* it satisfies $\phi = \phi + 2\pi R$, where R is related to the Luttinger parameter K as $R^2 = 1/(2\pi K)$. The importance of this restriction is discussed in the Appendix.

Let us consider first the XY case, *i.e.* $\Delta = 0$, and for convenience let us rotate by π the spins on every second site around the z axis in spin space, which simply amounts to an irrelevant change of the overall sign of the exchange term (in this case J and $-J$ lead to equivalent models).

Then it is convenient to write the spin operators in terms of spinless fermions ψ_n, through the so-called Jordan-Wigner transformation:

$$S_n^z = \psi_n^\dagger \psi_n - 1/2 \tag{6.59}$$

$$S_n^+ = e^{-i\alpha_n}\psi_n^\dagger\,, \qquad \alpha_n = \pi\sum_{j=0}^{n-1}\left(\psi_j^\dagger\psi_j\right)\,. \tag{6.60}$$

It is easy to show that these operators satisfy the $SU(2)$ algebra (6.1) provided $S = 1/2$ and the spinless fermions ψ_n are canonical, *i.e.* $\{\psi_n, \psi_{n'}^\dagger\} = \delta_{n,n'}$.

The Hamiltonian (6.56) can then be written as

$$H_{XY}^{latt} = J\sum_{n=1}^{N}\left(-\frac{1}{2}\left(\psi_n^\dagger\psi_{n+1} - \psi_n\psi_{n+1}^\dagger\right) - h\left(\psi_n^\dagger\psi_n - 1/2\right)\right)\,. \tag{6.61}$$

This problem can be readily solved by Fourier transforming

$$\tilde{\psi}_k = \frac{1}{\sqrt{N}}\sum_n \psi_n e^{-ikna} \tag{6.62}$$

where a is the lattice spacing and the momentum k is restricted to the first Brillouin zone, $k \,\epsilon\, (-\pi/a, \pi/a]$.

$$H_{XY}^{latt} = -J\sum_k \cos(k)\tilde{\psi}_k^\dagger\tilde{\psi}_k - h\sum_k \tilde{\psi}_k^\dagger\tilde{\psi}_k. \tag{6.63}$$

where we see that we have one band of fermions with dispersion $e(k) = -J\cos(k)$ and chemical potential $-h$. The ground state is obtained by filling all single particle states which have energies $e(k) < h$ as in Fig. 6.1

Normalizing the magnetization as $M = \frac{2}{N}\sum_n S_n^z = M(h)$, we see that the Fermi momentum is given by

$$k_F = \pm\frac{\pi}{2}(1 + M). \tag{6.64}$$

Since we are interested in the low energy properties of the model, we keep only the modes close to the Fermi surface (here consisting of two points) by restricting the sum in (6.63) to $|k \pm k_F| \le \Lambda$, with Λ an ultraviolet cutoff. This allows us to study the system at length scales larger than $1/\Lambda$.

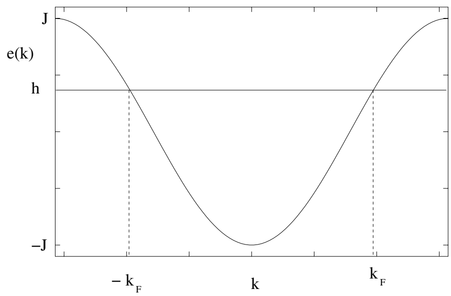

Fig. 6.1. Dispersion

Writing the fermions as $(x = na)$

$$\frac{\psi(x)}{\sqrt{a}} \approx e^{ik_F x}\psi_L(x) + e^{-ik_F x}\psi_R(x) \tag{6.65}$$

where ψ_R and ψ_L vary slowly with x in a scale of order $a > 1/\Lambda$, and contain the Fourier modes around $\pm k_F$ respectively, we obtain

$$H_{XY}^{cont} = iv \int dx[\psi_R^\dagger \partial_x \psi_R - \psi_L^\dagger \partial_x \psi_L] \tag{6.66}$$

where the Fermi velocity $v = \partial e(k)/\partial k|_{k=k_F} = Ja\sin(k_F)$ which we set to 1 in what follows. This is the Dirac Hamiltonian in $(1+1)$ dimensions. This means that the low energy theory for the XX case (*i.e.* $\Delta = 0$) corresponds to free fermions.

One can easily compute the fermion two point functions for right and left movers that are given by

$$\langle \psi_R(x,t)\psi_R^\dagger(0,0)\rangle = \frac{1}{2\pi a}\frac{1}{z} \tag{6.67}$$

$$\langle \psi_L(x,t)\psi_L^\dagger(0,0)\rangle = \frac{1}{2\pi a}\frac{1}{\bar{z}} \tag{6.68}$$

where $z = t + ix$, $\bar{z} = t - ix$.

From these correlators one can compute the one particle momentum distribution functions which show the characteristic Fermi liquid behaviour. We will see below that this behaviour is changed radically as soon as interactions are taken into account.

In order to treat the interactions that arise for $\Delta \neq 0$ it is more convenient to map the fermionic theory into an equivalent bosonic one, a procedure usually called bosonization.

It can be shown [19] that the fermionic theory described by (6.66) can be equivalently reformulated in terms of bosonic variables with Hamiltonian

$$H = \frac{1}{2}\int dx[(\partial_x\phi)^2 + (\partial_x\tilde{\phi})^2], \tag{6.69}$$

where ϕ is a scalar field and $\tilde{\phi}$ is defined in terms of the conjugate momentum $\Pi(x) = \partial_x\tilde{\phi}(x)$. Canonical commutation relations between ϕ and Π imply

$$[\phi(x), \tilde{\phi}(x')] = -\frac{i}{2}\mathrm{sign}(x - x') \tag{6.70}$$

while all other commutators are zero.

The key observation is that the fermion operators can be written in terms of the scalar field as

$$\psi_R(x) = \eta_R\frac{1}{\sqrt{2\pi a}} : e^{i\sqrt{4\pi}\phi_R(x)} : , \qquad \psi_L(x) = \eta_L\frac{1}{\sqrt{2\pi a}} : e^{-i\sqrt{4\pi}\phi_L(x)} \tag{6.71}$$

where $\eta_{R,L}$ are the so-called Klein factors which satisfy anticommutation relations $\{\eta_i, \eta_j\} = 2\delta_{ij}$. These Klein factors are operators which act on an auxiliary Hilbert space that can be chosen arbitrarily and this freedom is exploited to eliminate them from the effective theory (see below). The right and left components $\phi_{R,L}$ are defined in terms of the bosonic field and its dual as

$$\phi = \phi_R + \phi_L \qquad \tilde{\phi} = \phi_R - \phi_L \ . \tag{6.72}$$

The fields in the right hand side of (6.71) obey anticommutation rules, as can be easily verified using (6.70), and their two-point functions reproduce the free fermion results (6.68) (see the Appendix for details).

One can further show that the fermionic currents can be bosonized as

$$\begin{aligned} J_R &= :\psi_R^\dagger\psi_R:(x) = -\frac{i}{\sqrt{\pi}}\partial_z\phi_R(x), \\ J_L &= :\psi_L^\dagger\psi_L:(x) = \frac{i}{\sqrt{\pi}}\partial_{\bar{z}}\phi_L(x) \end{aligned} \tag{6.73}$$

where Klein factors do not appear here since $\eta_i^\dagger\eta_i = 1$ for $i = R, L$. In the following we will not include the Klein factors explicitly to simplify the notation. This is only possible whenever one can simultaneously diagonalize all the Klein operators which appear in a given problem, which is trivially the case for a single chain: In this case we have only two different Klein operators η_R and η_L and henceforth the only non-trivial products that could appear in the interaction terms are $t_{RL} \equiv \eta_R\eta_L$ and $t_{LR} = -t_{RL}$. We can then choose a basis of the Hilbert space where Klein operators act which diagonalizes t_{RL} and t_{LR} simultaneously and then forget about them. We discuss this issue in more detail in the case of N-leg ladders in Sect. 9.9 where the situation is a bit more complicated.

The interaction terms which arise when $\Delta \neq 0$ *i.e.*,

$$\delta H = \Delta \sum_{n=1}^{N} \left(S_n^z S_{n+1}^z \right) = \Delta \sum_{n=1}^{N} \left(\left(\psi_n^\dagger \psi_n - 1/2 \right) \left(\psi_{n+1}^\dagger \psi_{n+1} - 1/2 \right) \right) , \tag{6.74}$$

can be rewritten using (6.59) and (6.65) as

$$\delta H = \Delta \int dx \, [\rho(x) + (-1)^x M(x)] \, . \, \left[\rho(x+a) + (-1)^{x+a} M(x+a)\right] , \tag{6.75}$$

where

$$\rho(x) =: \psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L : \quad \text{and} \quad M(x) = \psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L . \tag{6.76}$$

Expanding up to first order in a and eliminating oscillatory terms one obtains

$$\delta H = \Delta \int dx \left(4 J_R J_L + J_R^2 + J_L^2 - \left((\psi_L^\dagger \psi_R)^2 + H.c. \right) \right) . \tag{6.77}$$

The first three terms which are quadratic in the currents are marginal in the renormalization group sense and can be easily handled using bosonization. The last one is irrelevant for $\Delta < 1$ so we postpone its analysis to a later stage. Quadratic interactions between currents arise in the so-called Thirring model and hence are usually termed "Thirring-like" terms.

The current-current terms are bosonized using (6.71) and (6.73) as

$$\delta H = \frac{1}{\pi} \Delta \int dx \left(4 \partial_x \phi_L \partial_x \phi_R - (\partial_x \phi_R)^2 - (\partial_x \phi_L)^2 \right) . \tag{6.78}$$

This term can be absorbed in (6.69) and the full bosonized XXZ Hamiltonian then reads

$$H = \frac{v}{2} \int dx \left[\frac{1}{K} (\partial_x \phi)^2 + K (\partial_x \tilde{\phi})^2 \right] , \tag{6.79}$$

where, up to first order in Δ, we have

$$K = 1 - \frac{2\Delta}{\pi} , \tag{6.80}$$

which provide the first term in the expansion of (6.58) for small Δ. The situation with the effective velocity v is less straightforward, as discussed in [20]. In this case one has to take into account the renormalization of the Fermi velocity due to the Δ interaction on the lattice before taking the continuum limit. In this way one gets to first order in Δ

$$v = 1 + \frac{2\Delta}{\pi} . \tag{6.81}$$

One can improve these results by using the exact Bethe Ansatz solution from which one can extract the exact values of v and K, as given in (6.58). The idea is to compare the asymptotics of the XXZ chain correlation functions obtained via the Bethe Ansatz solution in a finite volume L with that of the free boson defined by (6.79). It should be stressed that the relation between k_F and M (6.64) is not modified by the interactions. One further shows in this way that the bosonic field has to be compactified with a radius R given in terms of the Luttinger parameter K as $R^2 = 1/(2\pi K)$. This means that ϕ and $\phi + 2\pi R$ are identified at each point, and this leads to strong restrictions to the possible perturbations which could appear (see the Appendix).

We can now study the effects of the interactions on the low energy behaviour.

To this end, let us first compute the two point functions of the fermions for $\Delta \neq 0$. Using (6.71) and (6.79) one can easily show that (6.67) modifies to

$$\langle \psi_R(x,t)\psi_R^\dagger(0,0)\rangle = \frac{1}{2\pi a}\frac{1}{z^{2d}\bar{z}^{2\bar{d}}} \tag{6.82}$$

where $d = (K+1/K+2)/8$,$\bar{d} = (K+1/K-2)/8$ and $K(\Delta)$ is given in (6.58). A similar expression is obtained for the left-handed fermions. One can already observe the drastic change in the exponents caused by the interactions.

The most dramatic effect of the interactions is the disappearance of the quasiparticle peak in the Fourier transformed Green function, with the consequent disappearance of the finite jump in the momentum distribution function.

More precisely, the spectral function at zero temperature which is defined as

$$\rho(q,\omega) \equiv -\frac{1}{\pi}\mathrm{Im}G^R(k_F+q,\omega) \ , \tag{6.83}$$

where $G^R(k,\omega)$ is the Fourier transformed retarded two point function

$$G^R(x,t) \equiv -i\Theta(t)\left\langle\left\{\psi_R(x,t),\psi_R^\dagger(0,0)\right\}\right\rangle \ , \tag{6.84}$$

can be computed to give

$$\rho(q,\omega) = -2\sin(2\pi D)\Gamma(1-2d)\Gamma(1-2\bar{d})|w-q|^{2d-1}|w+q|^{2\bar{d}-1} \ . \tag{6.85}$$

where $D = d + \bar{d}$ is the scaling dimension of the interacting fermion.

From this last expression one can obtain the single particle density of states by integrating over the momentum, which leads to a power law behaviour

$$N(\omega) \approx |w|^{2D-1} \tag{6.86}$$

instead of the delta function peak characteristic of a Fermi liquid.

One can also compute the momentum distribution which gives

$$n(k) \approx n(k_F) + \text{const. sign}(k - k_F)|k - k_F| + \cdots \tag{6.87}$$

instead of the Fermi liquid behaviour in which $n(k)$ presents a finite jump at k_F, showing again the radical difference between the low energy theory of the XXZ chain and a Fermi liquid.

Another crucial difference between a Fermi liquid and our present theory is that in the former the exponents that control the space decay of correlations are universal (in the sense that they do not depend on the interactions) while they do depend on the interactions in the latter case.

All these features have motivated the name of Luttinger liquid to describe this kind of systems [10].

As a final step, the bosonized expressions for the spin operators are obtained using (6.59), (6.60), (6.65) and (6.71) leading to

$$S_x^z \approx \frac{1}{\sqrt{2\pi}} \frac{\partial \phi}{\partial x} + a : \cos(2k_F x + \sqrt{2\pi}\phi) : + \frac{\langle M \rangle}{2} \,, \tag{6.88}$$

and

$$S_x^{\pm} \approx \; (-1)^x : \mathrm{e}^{\pm i\sqrt{2\pi}\tilde{\phi}} (b \cos(2k_F x + \sqrt{2\pi}\phi) + c) : \tag{6.89}$$

where we have rescaled $K \to 2K$ in what follows, so that the free fermion point now corresponds to $K = 2$. The colons denote normal ordering with respect to the groundstate with magnetization $\langle M \rangle$, which leads to the constant term in (6.88). The prefactor $1/2$ arises from our normalization of the magnetization to saturation values $\langle M \rangle = \pm 1$. The constants a, b and c are non-universal and can be computed numerically and in particular an exact expression for b has been proposed in [21] for $h = 0$.

As we mentioned above, the parameter K in (6.57) can be computed by solving a set of integral equations obtained in the Bethe ansatz solution [22]. The results obtained from them are summarized in the magnetic phase diagram for the XXZ-chain (Fig. 6.2). There are two gapped phases: A ferromagnetic one at sufficiently strong fields and an antiferromagnetic phase for $\Delta > 1$ at small fields. In between is the massless phase where the bosonized form (6.57) is valid [18].

The transition between the ferromagnetic commensurate phase and the massless incommensurate phase, which occurs on the line $h_{uc} = (1 + \Delta)J$, is an example of the Dzhaparidze-Nersesyan-Pokrovsky-Talapov, universality class [23, 24], *i.e.* for $\langle M \rangle \to 1$ the magnetization behaves as

$$(\langle M \rangle - M_c)^2 \sim h^2 - h_{uc}^2 \tag{6.90}$$

with here $M_c = 1$.

This transition, which is an example of a commensurate-incommensurate (C-IC) transition can be described in the bosonization language by noticing

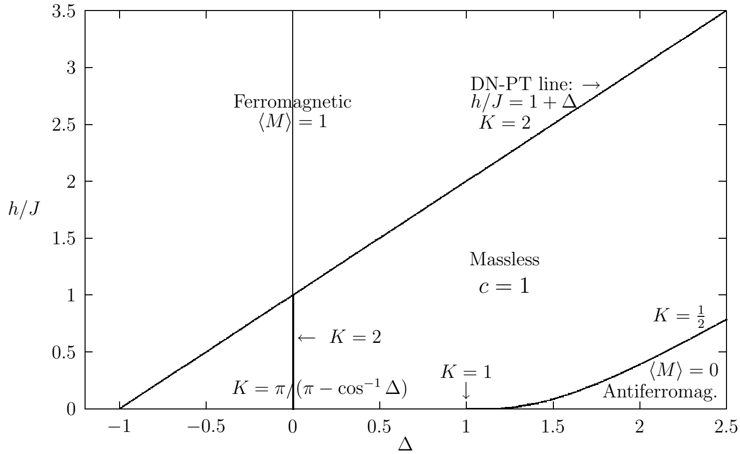

Fig. 6.2. Magnetic phase diagram of the XXZ-chain (6.56).

that for magnetic fields above saturation, ($h > h_{uc}$), one has to consider an additional operator which becomes commensurate for $\langle M \rangle = 1$. The Hamiltonian is then given by

$$H = H_0 + \int dx \cos \sqrt{2\pi}\phi(x) + h_{\text{eff}} \int dx \partial_x \phi \qquad (6.91)$$

where the last term corresponds to the interaction with the magnetic field in the bosonized language ($h_{\text{eff}} \propto h$) has no effect for $h > h_{uc}$ due to the presence of the gap [25]. The cos term which arises at $\langle M \rangle = 1$ is relevant and then responsible for the gap. By decreasing $h \to h_{uc}$ one can then drive the system into a massless regime and precisely at the transition point the Luttinger parameter takes the universal value $K = 2$ [23, 24].

The other transition line starts at $\Delta = 1$ and $h/J = 0$, *i.e.* at the $SU(2)$ point (see Sect. 9.3 for the study of this case using non-abelian bosonization). The Luttinger parameter takes the value $K = 1$ at this point and hence one has to include in the analysis of the low-energy dynamics the operator of dimension $2K$

$$\mathcal{O}(x) = \cos(\sqrt{8\pi}\phi(x)) \ , \qquad (6.92)$$

which is marginal at this point and becomes relevant for smaller K (bigger Δ). One can easily show how this operator arises by plugging the bosonized expression of S^z (6.88) in the Δ interaction term (6.74). This operator opens a gap in the spectrum via a Kosterlitz-Thouless transition [26] and from the Bethe Ansatz equations one readily obtains the characteristic stretched exponential decay for the gap for Δ slightly bigger than one:

$$\frac{h_c}{J} \sim 4\pi \mathrm{e}^{-\frac{\pi^2}{2\sqrt{2(\Delta-1)}}} \qquad \text{(for } \Delta \text{ slightly bigger than 1).} \qquad (6.93)$$

6.9.2 Thermodynamics and Correlations

We are now ready to analyze the thermodynamic properties of the XXZ chain in the low energy limit. Spin-spin correlation functions can be computed using (6.88, 6.89) together with the Hamiltonian (6.57) as well as (6.186, 6.189, 6.195) in the Appendix, with $g = 1/K$. One obtains in this way the following expressions for the equal time correlators (we set $m = 1$ hereafter):

$$\langle S^z_{x_1} S^z_{x_2} \rangle \approx \frac{\langle M \rangle^2}{4} + \frac{K}{4\pi^2} \frac{1}{|x_1 - x_2|^2} + \frac{a^2}{2} \frac{\cos(2k_F(x_1 - x_2))}{|x_1 - x_2|^K} \tag{6.94}$$

$$\langle S^+_{x_1} S^-_{x_2} \rangle \approx -\frac{b^2}{2} \frac{\cos((2k_F - \pi)(x_1 - x_2))}{|x_1 - x_2|^{K + \frac{1}{K}}} + (-1)^{(x_1 - x_2)} \frac{c^2}{|x_1 - x_2|^{\frac{1}{K}}} \tag{6.95}$$

where both staggered and non-staggered contributions are obtained.

From (6.94) we observe that for $\Delta > 0$, *i.e.* in the AF region, $K < 2$ and hence the staggered contribution dominates, signaling the expected tendency towards antiferromagnetic ordering. For $\Delta > 2$ instead, since $K > 2$, it is the non-staggered term that dominates at long distances, as expected in the ferromagnetic side. However, as expected in one dimension, there is no true long range order since the correlators decay slowly with a power law, which is called quasi-long range order. More importantly, it should be stressed that the power law decay is given by the Luttinger parameter K which is non-universal and depends on the microscopic details, such as the anisotropy Δ the magnetic field, etc.

Using the above expressions one can compute different thermodynamic properties such as the magnetic static susceptibility [27] and transport properties such as the dynamical susceptibility and thermal conductivity. These computations can be extended to finite (small) temperature by performing a conformal transformation which maps the plane (z) into the cylinder (ζ). This transformation compactifies the imaginary time direction via

$$z(\zeta) = \exp(2\pi\zeta/\beta) \;, \tag{6.96}$$

where $\beta = 1/T$.

Following [27], let us compute the magnetic susceptibility, which is defined as

$$\chi \equiv \frac{\partial M}{\partial h} = \beta \, \frac{Tr[(\sum_n S^z_n)^2 \, e^{-\beta H}]}{Tr\left[e^{-\beta H}\right]} - \beta \frac{Tr\left[(\sum_n S^z_n)\, e^{-\beta H}\right]^2}{\left[Tr\left[e^{-\beta H}\right]\right]^2} . \tag{6.97}$$

and hence

$$\chi = \beta \left(L \sum_n \langle S^z_n S^z_0 \rangle - M^2 \right) \tag{6.98}$$

Using the bosonized expression for the spin operators and noticing that after the integration the oscillating terms are eliminated, we are led to compute

$$\beta L \sum_n \langle S^z_n S^z_0 \rangle \rightarrow \beta \int_{-\infty}^{\infty} dx \, \langle S^z_x S^z_0 \rangle = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \, \langle \partial_x \phi(x) \partial_x \phi(0) \rangle \,, \quad (6.99)$$

Using (6.187), (6.188) one can easily compute the needed zero temperature correlations (recovering the Fermi velocity)

$$\langle \partial_x \phi_{R,L}(x,\tau) \partial_x \phi_{R,L}(0,0) \rangle = -\frac{K}{4\pi(v\tau \pm ix)^2} \,. \quad (6.100)$$

We can extend this result to finite (but small) temperatures by means of the conformal transformation (6.96), which leads to the replacement

$$v\tau \pm ix \rightarrow (v\beta/\pi) \sin(\pi \frac{v\tau \pm ix}{v\beta}) \quad (6.101)$$

in (6.100).

We are thus led to evaluate

$$\beta \int_{-\infty}^{\infty} dx \, \langle S^z_x S^z_0 \rangle = -\frac{K}{8v^2\beta} \int_{-\infty}^{\infty} dx \left(\frac{1}{\sin^2(\pi \frac{v\tau + ix}{v\beta})} + \frac{1}{\sin^2(\pi \frac{v\tau - ix}{v\beta})} \right) , \quad (6.102)$$

which can be easily done by using the following change of variables $u = \tan(\pi\tau/\beta); w = -i \tan(i\pi x/(v\beta))$.

We finally obtain

$$\chi = \frac{K}{\pi v} . \quad (6.103)$$

This result is valid for small temperatures and independent of T as it is expected from the scale invariance of the system.

By including the effects of the operator (6.92) which is irrelevant for $\Delta < 1$ and becomes marginal at the $SU(2)$ point as we already discussed, one can compute the next to leading term in the low temperature behavior of the susceptibility. One can do this by computing the two point correlator of the current in (6.99) using perturbation theory to include the perturbation term. For $1/2 < \Delta < 1$ one obtains a correction term proportional to $T^{4(K-1)}$ and for $\Delta < 1/2$ it takes the universal form T^2. In the $SU(2)$ case, $\Delta = 1$, the perturbation is marginally irrelevant and the correction term to the low temperature susceptibility is then logarithmic, $\propto \ln^{-1}(T_0/T)$ with T_0 a given constant. This result has been shown to agree quite well with the exact Bethe Ansatz result [27].

Notice that the susceptibility diverges when we approach the ferromagnetic point $\Delta \rightarrow -1$ because both v and K^{-1} vanish in this limit (see (6.58)). In the massive regime, which occurs for $\Delta > 1$, one obtains the expected exponential decay for $T \rightarrow 0$.

6.9.3 $SU(2)$ Point via Non-abelian Bosonization

For $\Delta = 1$ and $h = 0$, which corresponds to $K = 1$ (or $R = 1/\sqrt{2\pi}$) and $\langle M \rangle = 0$ (and hence $k_F = \pi/2$) one recovers the full $SU(2)$ spin symmetry. This can be observed *e.g.* in (6.94, 6.95), since they coincide at this particular point:

$$\langle S^z_{x_1} S^z_{x_2} \rangle \approx -\frac{1}{4\pi^2} \frac{1}{|x_1 - x_2|^2} + (-1)^{(x_1 - x_2)} \frac{a^2}{|x_1 - x_2|} \tag{6.104}$$

$$\langle S^+_{x_1} S^-_{x_2} \rangle \approx \frac{b^2}{|x_1 - x_2|} + (-1)^{(x_1 - x_2)} \frac{c^2}{|x_1 - x_2|} \tag{6.105}$$

For certain purposes it is more convenient to use non-abelian bosonization [28] and rewrite both the low energy Hamiltonian and the continuum expressions for the spin operators in this new language.

It can be shown that the scalar boson compactified at radius $R = 1/\sqrt{2\pi}$ is equivalent to the theory describing a $SU(2)$ group valued (matrix) field g with dynamics given by the Wess-Zumino-Witten (WZW) action [28]

$$\begin{aligned} S[g]_{WZW} = {} & \frac{k}{8\pi} \int d^2x \mathrm{tr}\left(\partial_\mu g \partial^\mu g^{-1}\right) \\ & + \frac{k}{12\pi} \int d^3y \epsilon_{ijk} \mathrm{tr}\left(g^{-1}\partial_i g g^{-1} \partial_j g g^{-1} \partial_k g\right) . \end{aligned} \tag{6.106}$$

where the trace is taken over the group indices and the so called level k equals 1 in the present case. This theory has been studied in [28] in the context of the non-Abelian bosonization of fermions and in [29] using conformal field theory techniques, where *e.g.* four point correlators were computed. See [30] for details.

The corresponding Hamiltonian can be written in the Sugawara form which is quadratic in the $SU(2)$ currents

$$H_{WZW} = \frac{1}{k+2} \int dx \left(\mathbf{J}_R \cdot \mathbf{J}_R + \mathbf{J}_L \cdot \mathbf{J}_L\right) \tag{6.107}$$

where $\mathbf{J}_{R,L} = \mathrm{tr}(\sigma g^{-1} \partial_{z,\bar{z}} g)$ and the spin operators can be compactly written as

$$\mathbf{S}_x \approx (\mathbf{J}_R + \mathbf{J}_L) + \mathrm{const}(-1)^x \mathrm{tr}(\sigma g) \tag{6.108}$$

The two formulation are related as follows

$$g \propto \begin{pmatrix} :\exp(i\sqrt{2\pi}\phi): & :\exp(-i\sqrt{2\pi}\tilde{\phi}): \\ -:\exp(i\sqrt{2\pi}\tilde{\phi}): & :\exp(-i\sqrt{2\pi}\phi): \end{pmatrix} \tag{6.109}$$

and

$$J^z_{R,L} = \pm\partial_{z,\bar{z}}\phi$$
$$J^+_L = :\exp(-i\sqrt{8\pi}\phi_L :)\ , \quad J^+_R =: \exp(-i\sqrt{8\pi}\phi_R) :\ . \qquad (6.110)$$

The marginally irrelevant perturbation (6.92) can be written in this language as the product of left and right handed currents $\mathbf{J}_R \cdot \mathbf{J}_L$.

At this point this (more complicated) formulation may appear unnecessary except for the fact that the expressions exhibit the $SU(2)$ invariance more naturally. However, the description of the $S = 1/2$ Heisenberg chain in terms of the level 1 WZW theory is crucial in the study of interacting systems, such as *e.g* the two leg Heisenberg ladder in the weak interchain coupling regime [31, 32]. In this case one can exploit the powerful machinery of CFT in two dimensions to study the low energy dynamics of these systems. This particular example is discussed in Sect. 9.7.

6.9.4 Modifications of the XXZ Chain

Using the formalism just developed one can study any modification of the XXZ chain provided that perturbation theory can be safely applied. We discuss now the case in which the exchange couplings J in the Hamiltonian (6.56) have a spatial periodicity of two sites (usually called dimerization) as a sample case, but other perturbations like next-nearest-neighbors, terms breaking XY symmetry, etc. can be treated similarly.

The Hamiltonian is given by

$$H^{latt}_{XXZ} = \sum_n J_n \left(\frac{1}{2} \left(S^+_n S^-_{n+1} + S^-_n S^+_{n+1} \right) + \Delta S^z_n S^z_{n+1} \right)\ , \qquad (6.111)$$

where $J_n = J(1 + (-1)^n \delta)$. For $\Delta = 0$ we can map it into a model of free fermions using the Jordan-Wigner tranformation (6.60)

$$H = \frac{J}{2} \sum_n \left((1-\delta)(\psi^\dagger_{2n}\psi_{2n+1} + H.c.) + (1+\delta)(\psi^\dagger_{2n+1}\psi_{2n+2} + H.c.) \right) \qquad (6.112)$$

Defining on even and odd sites the fermions $\chi_n = \psi_{2n}$ and $\xi_n = \psi_{2n+1}$ and Fourier transforming, one obtains a two by two Hamiltonian which can be diagonalized to give

$$H = J \sum_k \left(E_+\ \psi^{(+)\dagger}_{-k} \psi^{(+)}_k + E_-\ \psi^{(-)\dagger}_{-k} \psi^{(-)}_k \right) \qquad (6.113)$$

where $\psi^{(+)}_k$ and $\psi^{(-)}_k$ are defined in terms of the Fourier component of χ and ξ as

$$\psi^{(+)}_k = \frac{(1-\delta) + (1+\delta)e^{-ik}}{\sqrt{2}E_+}\ \chi_k + \frac{1}{\sqrt{2}}\ \xi_k\ ,$$
$$\psi^{(-)}_k = \frac{(1-\delta) + (1+\delta)e^{-ik}}{\sqrt{2}E_-}\ \chi_k + \frac{1}{\sqrt{2}}\ \xi_k\ . \qquad (6.114)$$

We have then two bands of fermions with dispersions given by

$$E_{\pm} = \pm\sqrt{(1+\delta^2+(1-\delta^2)\cos k)/2} \qquad (6.115)$$

which shows that a half filling there is a gap δ in the spectrum (see Fig. 6.3). Notice that the momentum k here is twice the momentum we have used in (6.62), due to the distinction between even and odd sites we made in going to the new variables χ and ξ.

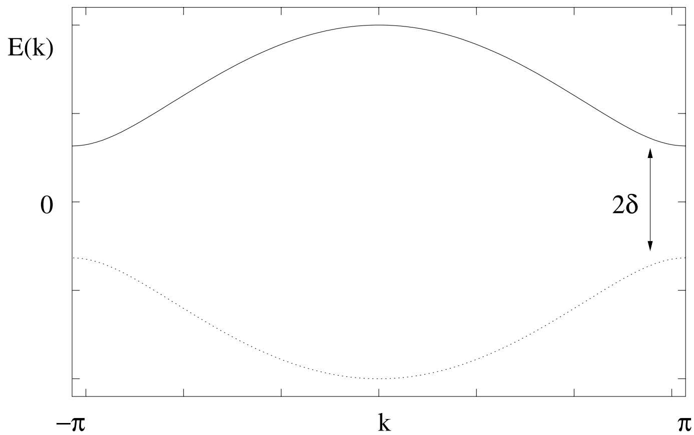

Fig. 6.3. Energy bands for the dimerized case.

The same model can be studied using bosonization now for arbitrary Δ but perturbatively in the dimerization δ. In this scheme one treats the term $J\delta\sum_n(-1)^n\left(\frac{1}{2}\left(S_n^+S_{n+1}^- + S_n^-S_{n+1}^+\right) + \Delta S_n^z S_{n+1}^z\right)$ as a perturbation which can be written using (6.88), (6.89). It is easy to show that a new term arises, which is of the form

$$O(x) = \cos\sqrt{2\pi}\phi \qquad (6.116)$$

which is relevant and is responsible for the opening of a gap. This operator could have been predicted by symmetry arguments, since once the translation symmetry is broken in the lattice, as it happens in the dimerized case, it is no longer forbidden to appear. This can be seen as follows: translation by one lattice site $x \to x+1$ implies that the chiral fermions in (6.65) transform as

$$\psi_R \to e^{ik_F}\psi_R, \qquad \psi_L \to e^{-ik_F}\psi_L \qquad (6.117)$$

and henceforth, for $k_F = \pi/2$, the bosonic field is transformed as $\phi \to \phi - \sqrt{\pi}/2$. Breaking of this symmetry then allows for a term like (6.116) to appear, which was otherwise forbidden. In the non Abelian $SU(2)$ formulation, the parity breaking operator is simply given by tr$g$.

The gap can be estimated by power counting to be of order $\approx \delta^{(1/(2-K/2))}$. Note that in the XY case (*i.e.* for $\Delta = 0$) $K = 2$ and we recover the free fermion result. In the next section we compute the RG equations for this effective theory.

6.9.5 RG Analysis of the Scalar Field Perturbed by Vertex Operators

We already used the renormalization group technique when treating the non-linear sigma model. The case of the scalar bosonic field with a vertex operator is much simpler. Following [8], the action is given by:

$$S = \frac{1}{2} \int dx \; dt \left[\frac{1}{K} \left(\partial_x \phi\right)^2 + \lambda \cos(\beta\phi) \right] , \qquad (6.118)$$

where $\beta = \sqrt{2\pi}$ corresponds to the dimerized case.

The scaling dimension of the operator $\cos(\beta\phi)$ is $\frac{K\beta^2}{4\pi}$ and the coupling λ has then the dimension $2 - \frac{K\beta^2}{4\pi}$ in order to have a dimensionless action.

Imagine now that, as before, we integrate over the fast degrees of freedom within the momenta shell $\Lambda - \delta\Lambda$ and Λ, or shifting from the scale L to $L + \delta L$. Since λ is a dimensional constant, it has to be rescaled accordingly. Simple dimensional analysis tells us that:

$$\frac{d\lambda}{dln(L)} = \left(2 - \frac{K\beta^2}{4\pi}\right) \lambda \qquad (6.119)$$

Of course one may anticipate that fluctuations can change this naïve scaling relation but to lowest order in the coupling constant we can keep this equation for describing the behavior of the system under renormalization group transformations. This is not, however, the end of the story. Let us assume that $\left(2 - \frac{K\beta^2}{4\pi}\right)$ is small *i.e.* we are close to the point where λ is marginal. In the process of integration, we define an effective partition function which we can define through the formal notation:

$$Z = Z_{\text{eff}}(1 - \lambda \int dx \; dt \langle \cos(\beta\phi(x,t)) \rangle +$$
$$\frac{\lambda^2}{2} \int dx_1 \; dx_2 \; dt_1 \; dt_2 \langle \cos(\beta\phi(x_1,t_1)) \cos(\beta\phi(x_2,t_2)) \rangle + ...) \qquad (6.120)$$

where the integration is taken over scales smaller than δL. The term in parenthesis can then be re-exponentiated and we can define our effective action in terms of the original one:

$$S_{eff} = S - \lambda \int dx \; dt \langle \cos(\beta\phi(x,t)) \rangle +$$
$$\frac{\lambda^2}{2} \int dx_1 \; dx_2 \; dt_1 \; dt_2 \langle \cos(\beta\phi(x_1,t_1)) \cos(\beta\phi(x_2,t_2)) \rangle + ... \qquad (6.121)$$

We then see that in the process of integrating out the degrees of freedom at small scales, we can have the merging of two vertex operators separated by a distance smaller than δL. In this operator product, there is certainly the term $\cos(2\beta\phi)$ which is present and contributes to the renormalization group equations, but can also be neglected to first order. There is however another term in the product expansion of the vertex operators:

$$e^{i\beta\phi(x)}e^{-i\beta\phi(x+\delta x)} \to -\frac{\beta^2\left(\partial_u\phi\right)^2}{|\delta x|^{\frac{K\beta^2}{2\pi}-2}} + ... \tag{6.122}$$

So the constant K gets a correction:

$$\begin{aligned}\frac{1}{K_{eff}} &= \frac{1}{K} + \lambda^2\beta^2 \int_{L}^{L+\delta L} \frac{d^2\delta x}{|\delta x|^{2-\epsilon}} \\ &= \frac{1}{K} + \frac{\lambda^2\beta^2}{\epsilon}2\pi((L+\delta L)^\epsilon - L^\epsilon)\end{aligned} \tag{6.123}$$

where $\epsilon = \frac{4-K\beta^2}{2\pi}$ is supposed to be small. From this result, and from (6.119) we can write the renormalization group equations to lowest order:

$$\begin{aligned}\frac{d\lambda}{dln(L)} &= \left(2 - \frac{K\beta^2}{4\pi}\right)\lambda + ... \\ \frac{dK}{dln(L)} &= -\frac{K^2\beta^2}{4\pi}\lambda^2 + ...\end{aligned} \tag{6.124}$$

which are known as the Kosterlitz renormalization group equations. The flow diagram for these equations is well known [8]. The flow is depicted in Fig. 6.4. In the vicinity of $\lambda = 0$, the line $\lambda = \lambda_c(K) = \left(\frac{K\beta^2}{8\pi} - 1\right)$ separates the regions of initial conditions that flow to weak coupling and strong coupling respectively. If $K > \frac{8\pi}{\beta^2}$ and $\lambda < \lambda_c(K)$, the large scale behavior of the system corresponds to a massless scalar field theory, while elsewhere the system presents a massive behavior with a finite correlation length.

6.9.6 Charge Degrees of Freedom: Hubbard and $t - J$ Models

The methods described in the previous sections can be extended to study systems including spin and charge degrees of freedom, provided they are Bethe ansatz solvable. Such is the case of the Hubbard model which is exactly solvable for arbitrary values of the on-site repulsion U, filling and magnetic field [33]. The exact solution can then be used to construct a low energy bosonized effective field theory [34–36] which can then be used to study perturbations of this model (see *e.g.* [14, 37]).

Here we present some aspects of the bosonization description of the Hubbard chain and its applications (see [15, 16, 37] and references therein).

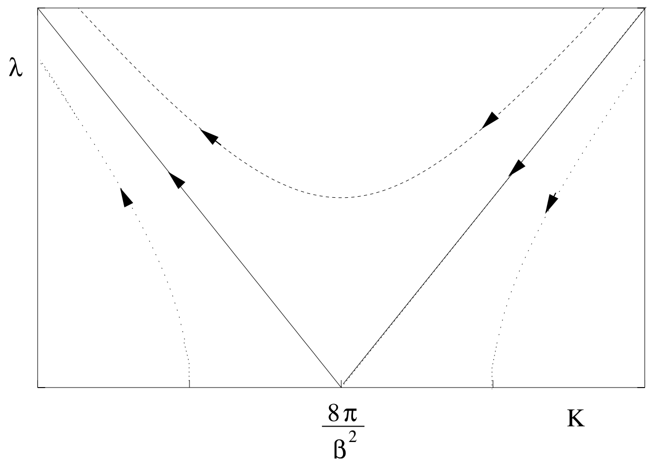

Fig. 6.4. Renormalization Group flow

The Hubbard model describes electrons hopping on a lattice which interact repulsively via an on-site Coulomb energy U with the lattice Hamiltonian given by

$$H = -t \sum_{n,\alpha} (c^\dagger_{n+1,\alpha} c_{n,\alpha} + H.c.) + U \sum_n c^\dagger_{n,\uparrow} c_{n,\uparrow} c^\dagger_{n,\downarrow} c_{n,\downarrow}$$
$$+\mu \sum_n (c^\dagger_{n,\uparrow} c_{n,\uparrow} + c^\dagger_{n,\downarrow} c_{n,\downarrow}) - \frac{h}{2} \sum_n (c^\dagger_{n,\uparrow} c_{n,\uparrow} - c^\dagger_{n,\downarrow} c_{n,\downarrow}) \ . \quad (6.125)$$

Here $c^\dagger_{n,\alpha}$ and $c_{n,\alpha}$ are electron creation and annihilation operators at site n, $\alpha = \uparrow, \downarrow$ the two spin orientations, h the external magnetic field and μ the chemical potential. As we already mentioned, this model has been exactly solved by Bethe Ansatz already in 1968 [33] but it took until 1990 for the correlation functions to be computed by combining Bethe Ansatz results with Conformal Field Theory (CFT) techniques [34].

Spin-charge separation is one of the important features of the Hubbard chain at zero magnetic field. Interestingly, it is no longer spin and charge degrees of freedom that are separated if an external magnetic field is switched on [34]. Nevertheless it has been shown that in the presence of a magnetic field, the spectrum of low energy excitations can be described by a semi-direct product of two CFT's with central charges $c = 1$ [34]. This in turn implies that the model is still in the universality class of the Tomonaga-Luttinger (TL) liquid and therefore allows for a bosonization treatment.

We proceed as before by setting $U = 0$ and writing the fermion operators as (now we have fermions with spin, and hence the number of equations is duplicated)

$$c_{n,\alpha} \to \psi_\alpha(x) \sim \mathrm{e}^{ik_{F,\alpha}x}\ \psi_{L,\alpha}(x) + \mathrm{e}^{-ik_{F,\alpha}x}\ \psi_{R,\alpha}(x)\ + \dots \quad (6.126)$$

$$= \mathrm{e}^{ik_{F,\alpha}x}\ \mathrm{e}^{-i\sqrt{4\pi}\phi_{L,\alpha}(x)} + \mathrm{e}^{-ik_{F,\alpha}x}\ \mathrm{e}^{i\sqrt{4\pi}\phi_{R,\alpha}(x)}\ + \dots\ , \quad (6.127)$$

where $k_{F,\alpha}$ are the Fermi momenta for up and down spin electrons, which are related to the filling and the magnetization as

$$k_+ = k_{F,\uparrow} + k_{F,\downarrow} = \pi n\ ; \qquad k_- = k_{F,\uparrow} - k_{F,\downarrow} = \pi\langle M\rangle\ , \quad (6.128)$$

The fields $\phi_{R,L,\alpha}$ are the chiral components of two bosonic fields, which bosonize the spin up and down chiral fermion operators $\psi_{R,L,\alpha}$, as in (6.71). The dots stand for higher order terms which have to be computed in order to reproduce the correct asymptotics of correlations obtained from the Bethe Ansatz solution. They take into account the corrections arising from the curvature of the dispersion relation due to the Coulomb interaction. The effects of band curvature due to interactions are also present in the case of the XXZ chain. However, in that case the effects are, for most practical purposes, negligible, since they lead in general to additional terms in the bosonization formulae which are strongly irrelevant operators. In the present case, though, these terms can be important since in some cases they could be relevant and should then be taken into account. For non-zero Hubbard repulsion U and magnetic field h, the low energy effective Hamiltonian corresponding to (6.125) written in terms of the bosonic fields $\phi_\uparrow$ and $\phi_\downarrow$ has a complicated form, mixing up and down degrees of freedom [36].

The crucial step to obtain a simpler bosonized Hamiltonian is to consider the Hamiltonian of a generalized (two component) TL model and identify the excitations of the latter with the exact Bethe Ansatz ones for the model (6.125), providing in this way a *non-perturbative* bosonic representation of the low energy sector of the full Hamiltonian (6.125). This program has been carried out in [36] and reviewed in [37].

The fixed point (*i.e.* neglecting all irrelevant terms) bosonized Hamiltonian can be written as

$$\sum_{i=c,s} \frac{u_i}{2} \int dx\ \left[(\partial_x\phi_i)^2 + (\partial_x\theta_i)^2 \right]\ , \quad (6.129)$$

where $\phi = \phi_R + \phi_L$ and $\theta = \phi_R - \phi_L$ and the new bosonic fields ϕ_c and ϕ_s are related to $\phi_\uparrow$ and $\phi_\downarrow$ through

$$\begin{pmatrix} \phi_c \\ \phi_s \end{pmatrix} = \frac{1}{\det Z} \begin{pmatrix} Z_{ss} & Z_{ss} - Z_{cs} \\ Z_{sc} & Z_{sc} - Z_{cc} \end{pmatrix} \begin{pmatrix} \phi_\uparrow \\ \phi_\downarrow \end{pmatrix}\ , \quad (6.130)$$

In these expressions Z_{ij}, $i,j = c,s$, are the entries of the dressed charge matrix Z taken at the Fermi points

$$Z = \begin{pmatrix} Z_{cc}\ Z_{cs} \\ Z_{sc}\ Z_{ss} \end{pmatrix}\ . \quad (6.131)$$

These matrix elements are solutions of a set of coupled integral equations obtained from the Bethe Ansatz [34] and depend on the Hubbard coupling U, the chemical potential μ and the magnetic field h. These parameters play a similar role as that played by K in the case of the XXZ chain.

At zero magnetic field, the matrix Z reduces to

$$Z(h=0) = \begin{pmatrix} \xi & 0 \\ \xi/2 & 1/\sqrt{2} \end{pmatrix} , \tag{6.132}$$

with $\xi = \xi(\mu, U)$. In this case we recover the expressions for the charge and spin fields for zero magnetic field

$$\phi_c = \frac{1}{\xi} \left(\phi_\uparrow + \phi_\downarrow\right) \ , \quad \phi_s = \frac{1}{\sqrt{2}} \left(\phi_\uparrow - \phi_\downarrow\right) \ , \tag{6.133}$$

where the compactification radius of the spin field (*i.e.* the parameter which indicates the period of ϕ_s, $\phi_s = \phi_s + 2\pi R_s$, $R_s = 1/\sqrt{2\pi}$) is fixed by the $SU(2)$ symmetry of the spin sector (it corresponds to the Luttinger parameter for the spin sector being $K_s = 1$). The radius for the charge field, on the other hand, depends on the chemical potential μ and the Coulomb coupling U.

One very important fact that we already mentioned is that for $h = 0$ the charge and spin degrees of freedom are completely decoupled, a phenomenon which is known as spin-charge separation. In particular, since the velocities for the two kinds of excitations are different, it is easy to verify that if one creates a particle (true electron) on the ground state, its constituents (spin and charge parts) will, after some time, be located in different points in space.

It should be noted that for $M \neq 0$, the fields arising in the diagonalized form of the bosonic Hamiltonian (6.129) are no longer the charge and spin fields.

For generic values of the parameters of the model (6.125), we can now write down for example the bosonized expressions for the charge density operator and for the z component of the spin operator

$$\begin{aligned} \rho(x) &= \psi_\uparrow^\dagger \psi_\uparrow(x) + \psi_\downarrow^\dagger \psi_\downarrow(x) \\ &= \frac{1}{\sqrt{\pi}} \partial_x \left(Z_{cc}\phi_c - Z_{cs}\phi_s\right) + a \ \sin[k_+ x - \sqrt{\pi}\left(Z_{cc}\phi_c - Z_{cs}\phi_s\right)] \\ &\quad \times \cos[k_- x - \sqrt{\pi}\left((Z_{cc} - 2Z_{sc})\phi_c - (Z_{cs} - 2Z_{ss})\phi_s\right)] \\ &\quad + b \sin(2k_+ x - \sqrt{4\pi}(Z_{cc}\phi_c - Z_{cs}\phi_s)) \ , \end{aligned} \tag{6.134}$$

$$\begin{aligned} 2S^z = \psi_\uparrow^\dagger \psi_\uparrow - \psi_\downarrow^\dagger \psi_\downarrow &= c \ \partial_x((Z_{cc} - 2Z_{sc})\phi_c - (Z_{cs} - 2Z_{ss})\phi_s) \\ &+ d \ \cos[k_+ x - \sqrt{\pi}(Z_{cc}\phi_c - Z_{cs}\phi_s)] \\ &\times \sin[k_- x - \sqrt{\pi}((Z_{cc} - 2Z_{sc})\phi_c - (Z_{cs} - 2Z_{ss})\phi_s)] \\ &- e \sin[2k_- x - \sqrt{4\pi}((Z_{cc} - 2Z_{sc})\phi_c - (Z_{cs} - 2Z_{ss})\phi_s)] \end{aligned} \tag{6.135}$$

where a, b, c, d, e are non-universal constants. Other operators can be constructed similarly and then correlations can be easily computed following similar lines as for the XXZ chain.

In the limit of large U, double occupancy will be forbidden and one can use perturbation theory in t/U to show that this Hamiltonian reduces to the so called $t-J$ model in this limit which for zero magnetic field reads

$$H_{t-J} = -t \sum_{n,\alpha} (c^\dagger_{n+1,\alpha} c_{n,\alpha} + H.c.) + J \sum_n \mathbf{S}_n \cdot \mathbf{S}_{n+1} \qquad (6.136)$$

where the operator $\mathbf{S}_n$ represents the spin of the electron at site n,

$$\mathbf{S}_n = c^\dagger_{n,\alpha} \frac{\sigma_{\alpha\beta}}{2} c_{n,\beta} \qquad (6.137)$$

with σ the Pauli matrices and the spin exchange constant is given by $J = t^2/U$.

In the case of zero field, $k_{F,\uparrow} = k_{F,\downarrow} = k_F$, and the expressions for the charge density and the S^z spin operators are simplified to

$$\begin{aligned} \rho(x) = & \frac{\xi}{\sqrt{\pi}} \partial_x \phi_c + a \ \sin(2k_F x - \sqrt{\pi}\ \xi\phi_c) \times \cos(\sqrt{2\pi}\phi_s) \\ & + b \sin(4k_F x - \sqrt{4\pi}\ \xi\phi_c)\ , \end{aligned} \qquad (6.138)$$

$$S^z = c\ \partial_x \phi_s + d\ \cos(2k_F x - \sqrt{\pi}\ \xi\phi_c) \times \cos(\sqrt{2\pi}\phi_s) + e \sin(\sqrt{8\pi}\phi_s) \qquad (6.139)$$

If one works at half-filling, which in this language means one electron per lattice site and hence $k_F = \pi/2$, there is an extra operator perturbing the charge sector which opens a charge gap even for arbitrarily small U. Then one can integrate out the charge degrees of freedom to recover the $S = 1/2$ Heisenberg chain studied before, which describes the Mott insulating phase of the Hubbard model.

After freezing the massive charge degrees of freedom, the spin operator reads

$$S^z = c\ \partial_x(\phi_s) + \mathit{const}\ \times (-1)^x \cos(\sqrt{2\pi}\phi_s)\ , \qquad (6.140)$$

where $\mathit{const} \propto \langle \cos(\sqrt{\pi}\ \xi\phi_c) \rangle$ and we recover the expression in (6.88) for $k_F = \pi/2$.

The $t-J$ model is not Bethe Ansatz solvable in general, but only at the specific point $J = 2t$ where it becomes supersymmetric [38]. At this point one can follow a similar procedure as described above to construct the bosonized low energy theory from the Bethe Ansatz solution.

6.9.7 Two-Leg Heisenberg Ladder

We have seen by using the NLSM approach that one should expect a spin gap in the spectrum of the two-leg Heisenberg ladder. In the present section we study the same problem using a different technique which further supports this conclusion.

We apply the combination of non-abelian bosonization techniques and the powerful machinery of conformal field theories in two dimensions to the case of a two leg $S = 1/2$ Heisenberg antiferromagnetic spin ladder following [31,32]. This is one of the simplest examples where the combination of these techniques shows its power by allowing for a complete analysis of the low energy dynamics.

The Hamiltonian is defined as

$$H^{latt}_{2-leg} = J\left(\mathbf{S}^1_n \cdot \mathbf{S}^1_{n+1} + \mathbf{S}^2_n \cdot \mathbf{S}^2_{n+1}\right) + J'\mathbf{S}^1_n \cdot \mathbf{S}^2_n, \tag{6.141}$$

where J, J' are the intrachain and interchain couplings respectively. We work in the weak interchain coupling limit $J' \ll J$, which allows us to apply the bosonization procedure described in Sect. 9.3 to each of the chains as if they were decoupled. We then treat the interchain couplings with the aid of (6.108) in perturbation theory.

The low energy limit Hamiltonian then takes the form

$$H^{cont}_{2-leg} = H^1_{WZW} + H^2_{WZW} + \lambda_1 \int dx \left((\mathbf{J}^1_R + \mathbf{J}^1_L) \cdot (\mathbf{J}^2_R + \mathbf{J}^2_L)\right) + \lambda_2 \int dx \left(\mathrm{tr}(\sigma g^1) \cdot \mathrm{tr}(\sigma g^2)\right) \tag{6.142}$$

where $\lambda_{1,2} \propto J'/J$.

The key observation here is that the free theory ($J' = 0$) corresponds to two $SU(2)_1$ WZW factors and this CFT theory can be conformally embedded into

$$SU(2)_1 \otimes SU(2)_1 \supset SU(2)_2 \otimes Z_2 \ , \tag{6.143}$$

where $SU(2)_2$ stands for the level 2 WZW theory and Z_2 corresponds to the Ising CFT.

This last equation does not indicate the complete equivalence of the theory on the r.h.s. with that on the l.h.s. What is true is that both theories have the same conformal central charge and all the primary fields of the theory on the l.h.s. are contained in the r.h.s theory. The idea is to try to map all the interaction terms into the new language, which in fact turns out to be possible in this case (though it is not generically true).

One can write the interaction terms in (6.142) using this embedding and the outcome is quite nice, since the two sectors are decoupled from each other, each of them with their respective mass terms.

There are two kinds of interaction terms in (6.142), the first being the current-current terms, which have the effect of renormalizing the effective Fermi velocity to first order apart from marginal terms. We then have the more relevant terms which are the product of the two WZW fields.

To study the effect of these relevant terms we use the conformal embedding mentioned above. The first observation is that the product of the WZW fields in the two $SU(2)_1$ sectors has scaling dimension 1 and should hence be writable in terms of dimension 1 operators in the Ising and $SU(2)_2$ WZW sectors. In this way, one obtains the following correspondence

$$\mathrm{tr}(\sigma g^1) \cdot \mathrm{tr}(\sigma g^2) = \mathrm{tr}(\Phi_{j=1}) - 3\,\epsilon\ , \tag{6.144}$$

which can be proved by comparing the operator product expansions of the operators on the left and right hand sides. In the above equation, the field $\Phi_{j=1}$ is the spin 1 field in the WZW theory $SU(2)_2$ and ϵ is the energy operator in the Ising sector, which can be described by one Majorana fermion.

We can then conclude that the Ising sector, being perturbed by the energy operator, has a mass m_1 proportional to J'/J.

This theory can be further simplified by noticing that the level 2 $SU(2)$ WZW theory can be equivalently described as three Majorana fermions. In this new language, the corresponding interaction term, $\mathrm{tr}(\Phi_{j=1})$ simply provides the mass m_2 for these Majorana fermions, which is different from m_1 and again proportional to J'/J. The ratio between the masses of the different Ising sectors has been fixed using Abelian bosonization in [32], showing that $m_1/m_2 = -3$.

The effective Hamiltonian can then be written as

$$\begin{aligned} H^{eff}_{2-leg} &= -\frac{i}{2}\left(\zeta_R \partial_x \zeta_R - \zeta_L \partial_x \zeta_L\right) - i m_1 \zeta_R \zeta_L \\ &+ \sum_{a=1}^{3} \left(-\frac{i}{2}\left(\xi^a_R \partial_x \xi^a_R - \xi^a_L \partial_x \xi^a_L\right) - i m_2 \xi^a_R \xi^a_L \right)\ , \end{aligned} \tag{6.145}$$

apart from marginal terms coming from the current-current interactions.

A similar result can be obtained using Abelian bosonization as in [32]. Different modifications of the two leg ladder considered here, as the inclusion of dimerization, extra diagonal couplings between the chains, etc. can be treated using the same formalism.

6.9.8 Higher Spin Chains: Non-abelian Bosonization

In the case of the Heisenberg antiferromagnet with higher values of the spin S one can still represent the spin variables in terms of fermions, which now carry an extra internal (color) index. The generalization of (6.137) for arbitrary S reads

$$\mathbf{S}_n = \sum_{i=1}^{2S} \sum_{\alpha,\beta=\uparrow,\downarrow} c^\dagger_{\alpha i n} \frac{\sigma_{\alpha\beta}}{2} c_{\beta i n}, \tag{6.146}$$

where $c_{\alpha i n}$ are fermionic variables with α the spin index, $i = 1, ..., 2S$ the extra color index and n the site index respectively. As before, $\sigma_{\alpha\beta}$ are the Pauli matrices.

In this case, non-abelian bosonization is more suitable to deal with the low energy theory. This approach has been first introduced in [39] (see also [40]). Here we follow the path-integral approach presented in [41], which is more suitable for our purposes.

In order to correctly represent the spin S chain, the physical states $|\text{phys}\rangle$ must satisfy at each lattice site the constraints

$$\begin{aligned} \sum_i c^\dagger_{\alpha i n} c_{\alpha i n} |\text{phys}\rangle &= 2S|\text{phys}\rangle \\ \sum_{i,j} c^\dagger_{\alpha i n} \tau^a_{ij} c_{\alpha j n} |\text{phys}\rangle &= 0, \end{aligned} \tag{6.147}$$

where τ^a are the generators of the $SU(2S)$ algebra. The first constraint imposes the condition that allows only one spin per site, whereas the second one states that the physical states must be color singlets.

The Heisenberg Hamiltonian (6.56) with spin operators satisfying (6.1) with $\Delta = 1$ and $h = 0$, can then be expressed as

$$H = -\frac{1}{2} \sum_n c^\dagger_{\alpha i n} c_{\alpha j n+1} c^\dagger_{\beta j n+1} c_{\beta i n} + \text{constant} \tag{6.148}$$

which has a local $SU(2S) \times U(1)$ local gauge invariance introduced by the parametrization (6.146). This quartic interaction can be rewritten by introducing an auxiliary field B as

$$H = \frac{1}{2} \sum_n (B^{ij}_{n,n+1} c^\dagger_{\alpha i n} c_{\alpha j n+1} + H.c. + \bar{B}^{ji}_{n+1,n} B^{ij}_{n,n+1}). \tag{6.149}$$

To obtain an effective low energy theory we perform a mean field approximation taking B as a constant $2S \times 2S$ matrix whereafter H can then be diagonalized. We then introduce the fluctuations around this mean solution, which we are able to integrate in a path-integral setup.

We write each color fermion as in (6.65)

$$\frac{\psi_{i\alpha}(x)}{\sqrt{a}} \approx e^{ik_F x} \psi_{L,i\alpha}(x) + e^{-ik_F x} \psi_{R,i\alpha}(x) \ , \tag{6.150}$$

with $k_F = \pi/2$ and expand the auxiliary field around its mean field value, keeping the fluctuations to first order in the lattice spacing, since we are interested in the low energy dynamics

$$B_{xy} = B_0 e^{aV_{xy}} \simeq B_0(1 + aV_{xy}), \tag{6.151}$$

We define the fields $A^1 \equiv \frac{1}{2}(V_{xy} - V_{xy}^\dagger)$ and $R_{xy} \equiv \frac{1}{2}(V_{xy} + V_{xy}^\dagger)$ in the algebra of $U(2S)$ which are the fluctuation fields which we have to integrate to obtain the effective low energy partition function. When substituted back into the Hamiltonian, the expansion (6.151) leads to a quadratic integral in R_{xy} which can be performed to give

$$\begin{aligned} H = B_0 &\left(-i\Psi_{R,i\alpha}^\dagger(\delta_{ij}\partial_x + A_{ij}^1)\Psi_{R,j\alpha} + i\Psi_{L,i\alpha}^\dagger(\delta_{ij}\partial_x + A_{ij}^1)\Psi_{L,j\alpha}\right) \\ &+ \frac{1}{4}\left(\Psi_{L,i\alpha}^\dagger\Psi_{R,j\alpha} - \Psi_{R,i\alpha}^\dagger\Psi_{L,j\alpha}\right)^2 , \end{aligned} \tag{6.152}$$

where the last term arises from the integration over the R field.

In order to implement the constraints (6.147) we first rewrite them in the continuum limit using (6.150). In terms of the continuum fermions the constraints read

$$\begin{aligned} \bar{\Psi}_{i\alpha}\gamma_0\Psi_{i\alpha}|\text{phys}\rangle &= 2S|\text{phys}\rangle \ , \\ \bar{\Psi}_{i\alpha}\gamma_0\tau_{ij}\Psi_{j\alpha}|\text{phys}\rangle &= 0 \ , \\ \bar{\Psi}_{i\alpha}\Psi_{j\alpha}|\text{phys}\rangle &= 0 \quad \text{for all } i,j \ , \end{aligned} \tag{6.153}$$

where $\Psi^\dagger = (\Psi_R, \Psi_L)$ and $\bar{\Psi} = \Psi^\dagger\gamma_0$.

The first two constraints are implemented by introducing a Lagrange multiplier A^0 in the Lie algebra of $U(2S)$, which together with A^1 in (6.152) provide the two space-time components of a gauge field in $U(2S)$. The third constraint is instead imposed with the use of the identity (see [41] for details)

$$\delta[\bar{\Psi}_{i\alpha}\Psi_{j\alpha}] = \lim_{\lambda_2\to\infty} e^{-\lambda_2 \int d^2x \ (\bar{\Psi}_{i\alpha}\Psi_{j\alpha})^2} \ . \tag{6.154}$$

After some algebra, the effective Lagrangian reads

$$L = \bar{\Psi}\gamma^\mu iD_\mu\Psi - \lambda_1(i\bar{\Psi}_i\gamma_5\Psi_j)^2 - \lambda_2(i\bar{\Psi}_i\Psi_j)^2, \tag{6.155}$$

where the covariant derivative is defined as $D_\mu = \partial_\mu - ia_\mu + B_\mu$, and we have decomposed, for later convenience, the $U(2S)$ A_μ field into a $U(1)$ field a_μ and a $SU(2S)$ field B_μ.

The Lagrangian can be further rewritten as

$$\begin{aligned} L = &\bar{\Psi}_{i\alpha}\gamma^\mu i(\partial_\mu - ia_\mu\delta_{ij}\delta_{\alpha\beta} + B_\mu^{ij}\delta_{\alpha\beta})\Psi_{j\beta} \\ &+4(\lambda_1 + \lambda_2)\mathbf{J}_R \cdot \mathbf{J}_L + (\lambda_1 + \lambda_2)j_R j_L \\ &-(\lambda_1 - \lambda_2)(\Psi_{Ri\alpha}^\dagger\Psi_{Lj\alpha}\Psi_{Rj\beta}^\dagger\Psi_{Li\beta} + H.c.), \end{aligned} \tag{6.156}$$

where

$$\mathbf{J}_{R,L} = \Psi^{\dagger}_{R,Li\alpha} \frac{\sigma_{\alpha\beta}}{2} \Psi_{R,Li\beta}$$
$$j_{R,L} = i\Psi^{\dagger}_{R,Li\alpha} \Psi_{R,Li\alpha} \tag{6.157}$$

are $SU(2)_{2S}$ and $U(1)$ currents respectively.

For $\lambda_1 = \lambda_2 = 0$ we are left with the theory of $2S$ Dirac fermions coupled to gauge fields in $U(1)$ and $SU(2S)$. Since these gauge fields have no dynamics, they act as Lagrange multipliers and it can be shown that the resulting theory corresponds to the fermionic realization of the coset model [42]

$$\frac{U(2S)}{U(1) \otimes SU(2S)_2} \equiv SU(2)_{2S} \tag{6.158}$$

as was already observed in [41]. The third term can be absorbed by a redefinition of the $U(1)$ gauge field a_μ.

The second and last terms in (6.156) can then be expressed as fields in the resulting WZW theory $SU(2)_{2S}$

$$\Delta\mathcal{L} = (\lambda_1 - \lambda_2) \left(\Phi^{(1/2)}_{\alpha\beta} \Phi^{(1/2)}_{\beta\alpha} + H.c. \right) + 4(\lambda_1 + \lambda_2) \mathbf{J}_R \cdot \mathbf{J}_L \tag{6.159}$$

where we have identified the spin 1/2 primary field of the $SU(2)_{2S}$ WZW theory, $\Phi^{(1/2)}$, in terms of its fermionic constituents

$$\Phi^{(1/2)}_{\alpha\beta} \equiv \Psi^{\dagger}_{R,i\alpha} \Psi_{L,i\beta} \tag{6.160}$$

which has conformal dimensions $d = \bar{d} = 3/(8(S+1))$. The first term in (6.159) corresponds then to the spin 1 affine primary $\Phi^{(1)}$ with conformal dimensions $d = \bar{d} = 1/(S+1)$, as can be seen after some simple algebra.

We can finally write

$$\Delta\mathcal{L} = -4\ (\lambda_1 - \lambda_2)\ tr\ \Phi^{(1)} + 4(\lambda_1 + \lambda_2) \mathbf{J}_R \cdot \mathbf{J}_L \tag{6.161}$$

For $S = \frac{1}{2}$ we recover the effective model we derived in Sect. 9.3. In this case, the first term in (6.161) is proportional to the identity operator and the second is marginally irrelevant since $\lambda_1 + \lambda_2$ is positive and gives the well known logarithmic corrections to correlators.

For higher spins, we have to consider the interaction term (6.161) and we also have to include all other terms which are radiatively generated. We then need the operator product expansion (OPE) coefficients among the different components of $\Phi^{(1)}$ which have been computed in [43]. The OPE coefficients are non-vanishing if the so called "Fusion Rules" are non-vanishing. In the level k $SU(2)$ WZW theory they are given by [44]

$$\Phi^{(j)}_{m,\bar{m}} \times \Phi^{(j')}_{m',\bar{m}'} = \sum_{n=|j-j'|}^{\min(j+j',k-j-j')} \Phi^{(n)}_{m+m',\bar{m}+\bar{m}'} \tag{6.162}$$

We can now make use of the following equivalence [43, 45]

$$SU(2)_k \equiv Z_k \otimes U(1) \tag{6.163}$$

We will exploit this equivalence to derive an effective low energy action for the spin S Heisenberg chain. Indeed, it was shown in [43] that the primary fields of the $SU(2)_k$ WZW theory are related to the primaries of the Z_k-parafermion theory and the $U(1)$ vertex operators. They are connected by the relation

$$\Phi^{(j)}_{m,\bar{m}}(z,\bar{z}) = \phi^{(2j)}_{2m,2\bar{m}}(z,\bar{z}) : e^{\frac{i}{\sqrt{2S}}(m\phi_R(z)+\bar{m}\phi_L(\bar{z}))} :, \tag{6.164}$$

where the Φ fields are the invariant fields of the $SU(2)_k$ WZW theory, the ϕ fields are the Z_k parafermion primaries and ϕ_R and ϕ_L are the holomorphic and antiholomorphic components of a compact massless free boson field. In the same way, the currents are related as

$$\begin{aligned} J^+_R(z) &= (2S)^{1/2}\psi_1(z) : \exp\left(\frac{i}{\sqrt{2S}}\phi_R(z)\right) : \ , \\ J^z_R(z) &= (2S)^{1/2}\partial_z\phi_R(z) \end{aligned} \tag{6.165}$$

where $J^\pm_R = J^x_R \pm iJ^y_R$ and ψ_1 is the first parafermionic field. (A similar relation holds for the left-handed currents).

Using this equivalence we can express the relevant perturbation term (6.161) in the new language as

$$\begin{aligned} \Delta\mathcal{L} = -4(\lambda_1-\lambda_2)\Big(\phi^{(2)}_{0,0} &+ \phi^{(2)}_{2,-2} : e^{\frac{i}{\sqrt{2S}}(\phi_R(z)-\phi_L(\bar{z}))} : \\ &+\phi^{(2)}_{-2,2} : e^{-\frac{i}{\sqrt{2S}}(\phi_R(z)-\phi_L(\bar{z}))} :\Big) \\ +4S(\lambda_1+\lambda_2)&\Big(\psi_1\bar{\psi}^\dagger_1 : e^{\frac{i}{\sqrt{2S}}(\phi_R(z)-\phi_L(\bar{z}))} : +H.c.\Big) , \end{aligned} \tag{6.166}$$

where we absorbed the derivative part of the $U(1)$ field coming from (6.165) into a redefinition of the constant in front of the unperturbed Lagrangian. The first term corresponds to the first "thermal" field of the parafermion theory, $\phi^{(2)}_{0,0} = \epsilon_1$, with conformal dimensions $d = \bar{d} = 1/(1+S)$, while the second and third terms correspond to the $p = 2$ disorder operator in the PF theory, $\phi^{(2)}_{2,-2} = \mu_2$ and its adjoint $\phi^{(2)}_{-2,2} = \mu^\dagger_2$ with dimensions $d_2 = \bar{d}_2 = (S-1)/(2S(S+1))$. It is assumed that all the operators which are radiatively generated have to be included in the complete effective theory.

We use now the fact that the Z_{2S} PF theory perturbed by its first thermal operator ϵ_1 flows into a massive regime irrespectively of the sign of the coupling [46]. Assuming that, as for the Z_2 case, due to the sign of the coupling $\lambda_1 - \lambda_2$ in (6.166) the theory is driven into a low temperature ordered phase, we have that vacuum expectation values (v.e.v.'s) of disorder operators μ_j, vanish for $j \neq 2S \bmod(2S)$ as well as v.e.v.'s of the parafermionic fields

$\langle\psi_k\bar{\psi}_k^\dagger\rangle = 0$, for $2k \neq 2S \bmod(2S)$. This will be important in the computation of spin-spin correlation functions below.

Since the parafermionic sector is massive, the effective theory for large scales can be obtained by integrating out these degrees of freedom. One can obtain then the most general effective action for the remaining $U(1)$ field, by including all the vertex operators which are invariant under the symmetry $Z_{2S} \times \tilde{Z}_{2S}$ [43]

$$\phi_R \to \phi_R - \frac{\sqrt{2}\pi m}{\sqrt{S}} \ ; \ \phi_L \to \phi_L - \frac{\sqrt{2}\pi n}{\sqrt{S}} \qquad (6.167)$$

with $m, n \in \mathbb{Z}$, which is preserved after the integration of the massive parafermions.

One obtains in this way the effective action for the remaining $U(1)$ theory

$$Z_{\text{eff}} = \int \mathrm{d}\phi \ \exp\Bigg(- \int K_S(\partial_\mu\phi)^2 + \alpha_S \int \cos(\sqrt{\frac{S}{2}}(\phi_R - \phi_L)) + \beta_S \int \cos(\sqrt{2S}(\phi_R - \phi_L)) + \cdots \Bigg) , \qquad (6.168)$$

for S integer while α_S vanishes for S half integer. Here the dots indicate irrelevant fields corresponding to higher harmonics of the scalar field and K_S is an effective constant arising from the OPE of vertex and parafermionic operators in the process of integration of the massive degrees of freedom.

Using the generalization of (6.65) to the case with $2S$ colors together with (6.146), (6.157) and (6.160) we can write the continuum expression of the original spin operator $\mathbf{S}(x)$ as

$$\mathbf{S}(x) = \mathbf{J}_R + \mathbf{J}_L + \text{const} \ (-1)^x \text{tr}(\frac{\sigma}{2}(\Phi^{(1/2)} + \Phi^{(1/2)\dagger})) \ , \qquad (6.169)$$

which is the generalization of (6.108) for arbitrary spin S.

Let us study the behavior of the spin-spin correlation function at large scales, to see whether the system has a gap or not. In the new language of (6.169), these correlators have a staggered and a non-staggered part which correspond respectively to current-current correlators and correlators of the components of the fundamental field $\Phi^{(1/2)}$.

Let us focus on the staggered part of the S^zS^z correlator: Since our original $SU(2)$ WZW model is perturbed, correlation functions of the fundamental field will contain supplementary operators coming from the OPE of the product of $\Phi^{(1/2)}$ and the perturbing fields. With the help of the fusion rules (6.162) it is easy to see that, for example, the effective alternating

z-component of the spin operator containing the scalar field will be given by:

$$\sum_{k \leq 2S,\ k \text{ odd}} a_k\ \mu_k : e^{\frac{ik}{2\sqrt{2S}}(\phi_R(z) - \phi_L(\bar{z}))} : + H.c., \tag{6.170}$$

where only odd k fields appear in the sum.

For S half-integer, the operator $\Phi^{(S)}$ is present in (6.170), and we can easily check that, (since μ_{2S} corresponds to the identity), this operator is simply given by

$$e^{\frac{iS}{\sqrt{2S}}(\phi_R - \phi_L)} + H.c.$$

The other operators in the series contain parafermionic disorder operators whose correlators will decay exponentially to zero at large scales. Thus, considering only the Gaussian part of (6.168), we can show that the spin correlation functions at large scales behave like:

$$\begin{aligned} \langle S_z(x) S_z(y) \rangle &\sim (-1)^{(x-y)} |x-y|^{-2SK_S} \\ \langle S_+(x) S_-(y) \rangle &\sim (-1)^{(x-y)} |x-y|^{-1/(2SK_S)} \end{aligned} \tag{6.171}$$

The fact that the $SU(2)$ symmetry is unbroken at all scales fixes then the value of K_S to be

$$K_S = 1/(2S) \tag{6.172}$$

For this value of K_S one can show that the perturbing operator with coupling β_S in (6.168) is marginally irrelevant (remember that $\alpha_S = 0$ in the half-integer case).

We conclude then that the large scale behavior of half-integer spin chains is given by the level 1 $SU(2)$ WZW model with logarithmic corrections as for the spin 1/2 chain.

Let us consider now integer spins S. Since the series (6.170) for the effective spin operator contains only half-integer spins j (odd k's), all the operators in the series will contain non-trivial parafermionic operators. Then all the terms in the spin-spin correlation function will decay exponentially to zero with the distance indicating the presence of a gap in the excitation spectrum, thus confirming Haldane's conjecture.

6.9.9 N-Leg Ladders in a Magnetic Field: Gap for Non-zero Magnetization

Another interesting situation is the one of antiferromagnetic spin ladders which we have already studied in Sect. 7 using NLSM techniques in the $SU(2)$ symmetric case. The Hamiltonian for coupled XXZ chains in the presence of a magnetic field is a generalization of that presented in (6.141) [18]

$$H^{latt}_{N-ladder} = \sum_{a=1}^{N} H^{(a)}_{XXZ} + J' \sum_{n,a=1}^{a=N} \mathbf{S}^a_n \cdot \mathbf{S}^{a+1}_n - h \sum_{i,n} S^{a,z}_n \,, \tag{6.173}$$

where $H^{(a)}_{XXZ}$ is given by an expression like (6.56) for each chain labeled by a.

For $J' = 0$ one can map the low energy sector of each XXZ chain into a bosonic field theory as described in Sect. 9. One obtains in this way an effective description which consists in a collection of identical Hamiltonians like (6.57), with N bosonic fields, ϕ_a, describing the low energy dynamics of chain a, $a = 1, \cdots, N$. The interchain exchanges give rise to perturbation terms which couple the fields of the different chains.

In the case in which many chains are considered, one has to introduce as many different Klein factors as the number of chains considered for both right and left components, η^a_R, η^a_L, $a = 1, ..., N$, to ensure the correct commutation relations between spin fields (see (6.71) for the case of a single chain). In the present case, the interactions contain generically products of four Klein factors of the form

$$t_{ijkl} \equiv \eta_i \eta_j \eta_k \eta_l \tag{6.174}$$

where the subindices here indicate the pair index (α, a), with $\alpha = R, L$ and a the chain index. One can easily show using the Klein algebra, $\{\eta_i, \eta_j\} = 2\delta_{ij}$, that $t^2 = 1$ when all indices are different and then these operators have eigenvalues ± 1. As discussed in [15], one could get rid of the t operators which appear in the interaction terms (and hence bosonize completely the problem) provided one can simultaneously diagonalize all the operators like (6.174) appearing in a given situation. This in turn can be done if all these operators are mutually commuting, which has to be studied for each case separately. In the present situation this can be easily shown by noticing that interchain interactions between a and b chains contain products of the form

$$\eta^a_R \eta^a_L \eta^b_R \eta^b_L \tag{6.175}$$

with $a \neq b$, $a, b = 1, ..., N$ and using the algebra of the Klein factors one can show that they are all mutually commuting.

After a careful RG analysis, one can show that at most one degree of freedom, given by the combination of fields $\phi_D = \sum_a \phi_a$, remains massless. The large scale effective action for the ladder systems is then given again by a Hamiltonian (6.57) for ϕ_D and the perturbation term

$$H_{pert} = \lambda \int dx \;\cos(2N k_F x + \sqrt{2\pi}\phi_D) \,, \tag{6.176}$$

where $k_F = (1 + \langle M \rangle)\pi/2$ is related to the total magnetization $\langle M \rangle$.

The key point if to identify the values of the magnetization for which the perturbation operator (6.176) can play an important rôle. In fact, this operator is commensurate at values of the magnetization given by

$$N/2(1-\langle M\rangle) \in \mathbb{Z}\,, \tag{6.177}$$

otherwise the integral over x will make this term vanish due to the fast oscillations of the phase factor since the continuum fields are slowly varying.

If this operator turns out to be also relevant in the RG sense (this depends on the parameters of the effective Hamiltonian (6.57), the model will have a finite gap, implying a plateau in the magnetization curve.

Let us see how this condition can be obtained: in the weak-coupling limit along the rungs, $J' \ll J$, the bosonized low-energy effective Hamiltonian for the N-leg ladder reads

$$\begin{aligned} H^{cont}_{N-ladder} = \int \mathrm{d}x \Bigg[\frac{1}{2}\sum_{a=1}^{N} \left(v_a K_a \left(\partial_x \tilde{\phi}_a(x)\right)^2 + \frac{v_a}{K_a}\left(\partial_x \phi_a(x)\right)^2 \right) \\ + \lambda_1 \sum_{a,b} \left(\partial_x \phi_a(x)\right)\left(\partial_x \phi_b(x)\right) \\ + \sum_{a,b} \Big\{ \lambda_2 : \cos(2(k_F^a + k_F^b)x + \sqrt{2\pi}(\phi_a + \phi_b)) : \\ + \lambda_3 : \cos\left(2(k_F^a - k_F^b)x + \sqrt{2\pi}(\phi_a - \phi_b)\right) : + \lambda_4 : \cos\left(\sqrt{2\pi}(\tilde{\phi}_a - \tilde{\phi}_b)\right) : \Big\} \Bigg]\,, \end{aligned} \tag{6.178}$$

where only the most relevant perturbation terms are kept. The four coupling constants λ_i essentially correspond to the coupling J' between the chains: $\lambda_i \sim J'/J$. In arriving to the Hamiltonian (6.178) we have discarded a constant term and absorbed a term linear in the derivatives of the free bosons into a redefinition of the applied magnetic field. For simplicity we have used here periodic boundary conditions (PBC's) along the transverse direction.

Note that the λ_2 and λ_3 perturbation terms contain an explicit dependence on the position (in the latter case this x-dependence disappears for symmetric configurations with equal k_F^i). Such operators survive in passing from the lattice to the continuum model, assuming that the fields vary slowly, only when they are commensurate. In particular, the λ_2 term appears in the continuum limit only if the oscillating factor $\exp(i2x(k_F^i + k_F^j))$ equals unity. If the configuration is symmetric, this in turn happens only for zero magnetization (apart from the trivial case of saturation).

Let us describe this in some detail for the case of the three leg ladder, $N = 3$. In this case we first diagonalize the Gaussian (derivative) part of the Hamiltonian by the following change of variables in the fields:

$$\psi_1 = \frac{1}{\sqrt{2}}\left(\phi_1 - \phi_3\right), \psi_2 = \frac{1}{\sqrt{6}}\left(\phi_1 + \phi_3 - 2\phi_2\right), \psi_D = \frac{1}{\sqrt{3}}\left(\phi_1 + \phi_2 + \phi_3\right). \tag{6.179}$$

In terms of these fields the derivative part of the Hamiltonian can be written as:

$$\bar{H}_{\mathrm{der.}} = \frac{vK}{2} \int \mathrm{d}x \left[(1+a) \left(\partial_x \psi_D(x)\right)^2 + (1-b) \left(\left(\partial_x \psi_1(x)\right)^2 + \left(\partial_x \psi_2(x)\right)^2 \right) \right] \tag{6.180}$$

where $a = J'K/J = 2b$. We can now study the large-scale behaviour of the effective Hamiltonian (6.178) where we assume all k_F^i equal due to the symmetry of the chosen configuration of couplings. Let us first consider the case when the magnetization $\langle M \rangle$ is non-zero. In this case only the λ_3 and λ_4 terms are present. The one-loop RG equations are:

$$\begin{aligned} \frac{dK}{d\ln(L)} &= -2K^2 \ \lambda_3^2 + 2\lambda_4^2 \\ \frac{d\lambda_3}{d\ln(L)} &= \left(2 - \frac{K}{(1-b)}\right) \lambda_3 - \pi\lambda_3^2 \\ \frac{d\lambda_4}{d\ln(L)} &= \left(2 - \frac{(1-b)}{K}\right) \lambda_4 - \pi\lambda_4^2 \,. \end{aligned} \tag{6.181}$$

It is important to notice that only the fields ψ_1 and ψ_2 enter in these RG equations, since the perturbing operators do not contain the field ψ_D. The behaviour of these RG equations depends on the value of K. The main point is that always one of the two λ perturbation terms will dominate and the corresponding cosine operator tends to order the associated fields. This gives a finite correlation length in correlation functions containing the fields ψ_1 and ψ_2 (or their duals). For example, for $\Delta \leq 1$ we have that $K > 1$ since $\langle M \rangle \neq 0$. Then, from (6.181) one can easily see that the dominant term will be the λ_4 one. This term orders the dual fields associated with ψ_1 and ψ_2. Then, the correlation functions involving these last fields decay exponentially to zero. In either case, the field ψ_D remains massless. In the case of open boundary conditions the situation is similar and again it is the diagonal field the one which stays generically massless, in spite of the asymmetry of the Gaussian part of the action.

A more careful analysis of the original Hamiltonian shows that this diagonal field will be coupled to the massive ones only through very irrelevant operators giving rise to a renormalization of its Luttinger parameter K. However, due to the strong irrelevance of such coupling terms these corrections to K are expected to be small, implying that its large-scale effective value stays close to the zero-loop result.

At the values of the magnetization where this operator is commensurate, the field ψ_D can then undergo a K-T transition to a massive phase, indicating the presence of a plateau in the magnetization curve. An estimate of the value of J' at which this operator becomes relevant can be obtained from its scaling dimension. In the zero-loop approximation and for $\Delta = 1$ one then obtains $J'_c \approx 0.09J$ for the $\langle M \rangle = 1/3$ plateau at $N = 3$.

Appendix: The Scalar Boson in 2D, a $c = 1$ Conformal Field Theory

Primary Field Content and Correlators

The action for the scalar Euclidean boson is

$$S(\phi) = \frac{g}{2} \int d^2x \, (\partial_\mu \phi)^2 \tag{6.182}$$

and in condensed matter applications g is related to the Luttinger parameter as $g = 1/K$ and hence contains the information about the interactions.

This action is invariant under constant translations of the field

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha \ , \tag{6.183}$$

with the corresponding conserved current $J_\mu(x) = \partial_\mu \phi(x)$. There exists another (trivially) conserved current $\tilde{J}_\mu(x) = \epsilon_{\nu\mu} \partial_\nu \phi(x)$ (usually referred to as "topological" current).

The corresponding Hamiltonian reads

$$H = \frac{1}{2} \int dx \left(\frac{1}{g} \, \Pi(x)^2 + g \, (\partial_x \phi)^2 \right) \ , \tag{6.184}$$

where the wave propagation velocity has been set to 1 and the canonical conjugate momentum $\Pi \equiv \delta \mathcal{L} / \delta \dot{\phi} = g \dot{\phi}$. The dual field $\tilde{\phi}$ which is usually defined for convenience, since it allows to write certain fields in a local way, is related to Π as $\partial_x \tilde{\phi} = \Pi$. One can eliminate g from (6.184) by making a canonical transformation

$$\phi' = \sqrt{g} \phi \ , \qquad \Pi' = \frac{1}{\sqrt{g}} \Pi \ . \tag{6.185}$$

The propagator is then given by

$$\Delta(z, \bar{z}; w, \bar{w}) \equiv \langle 0 | \phi'(z, \bar{z}) \phi'(w, \bar{w}) | 0 \rangle = -\frac{1}{4\pi} \log m^2 |z - w|^2 \tag{6.186}$$

where $z = v\tau + ix$, $\bar{z} = v\tau - ix$ and m is a small mass which has been added as an infrared regulator. Ultraviolet divergences are naturally regulated in the problems we will be interested in by the lattice constant a. We will drop the primes in the scalar fields from now on, but the reader should keep in mind (6.185).

From this correlator one can read the chiral parts ($\phi(z, \bar{z}) = \phi_R(z) + \phi_L(\bar{z})$)

$$\langle \phi_R(z) \phi_R(w) \rangle = -\frac{1}{4\pi} \log m(z - w) \tag{6.187}$$

and

$$\langle \phi_L(\bar{z})\phi_L(\bar{w})\rangle = -\frac{1}{4\pi}\log m(\bar{z}-\bar{w}) \ . \tag{6.188}$$

In terms of the chiral components the dual field reads $\tilde{\phi} = \phi_R(z) - \phi_L(\bar{z})$.

Taking derivatives from (6.187) one obtains

$$\langle \partial_z\phi_R(z)\partial_w\phi_R(w)\rangle = -\frac{1}{4\pi}\frac{1}{(m(z-w))^2} \tag{6.189}$$

and similarly for the anti-holomorphic components.

The energy-momentum tensor for the free massless boson is

$$T_{\mu\nu} =: \left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial_\rho\phi\right) : , \tag{6.190}$$

where the dots : : denote normal ordering defined by substracting the singular part of the product when the arguments coincide.

Its holomorphic component reads

$$T \equiv -2\pi T_{zz} = -2\pi : \partial\phi_R\partial\phi_R := \lim_{z\to w}\left(\partial\phi_R(z)\partial\phi_R(w) - \langle\partial\phi_R(z)\partial\phi_R(w)\rangle\right) \tag{6.191}$$

and similarly for the anti-holomorphic component $\bar{T}(\bar{z})$.

The two point correlators of the energy-momentum tensor components can be easily computed using (6.189) and (6.191) to give

$$\langle T(z)\ T(w)\rangle = \frac{c/2}{(m(z-w))^4} \ , \qquad \langle \bar{T}(\bar{z})\ \bar{T}(\bar{w})\rangle = \frac{c/2}{(m(\bar{z}-\bar{w}))^4}, \tag{6.192}$$

with the numerical constant $c = 1$. These expressions define the central charge of the model. Indeed, it can be shown that the modes of the energy momentum tensor components T and $\bar{T}$ satisfy respective Virasoro algebras with central charge $c = 1$.

One usually defines vertex operators in terms of the chiral components of the field ϕ as

$$\begin{aligned} V_{\alpha,\bar{\alpha}}(z,\bar{z}) &=: \exp\left(i\alpha\phi_R(z) + i\bar{\alpha}\phi_L(\bar{z})\right) := \\ &: \exp\left(i\frac{(\alpha+\bar{\alpha})}{2}\phi(z,\bar{z}) + i\frac{(\alpha-\bar{\alpha})}{2}\tilde{\phi}(z,\bar{z})\right) : , \end{aligned} \tag{6.193}$$

for arbitrary real numbers α and $\bar{\alpha}$, where normal ordering is defined as usual and for the vertex operators we can write

$$: e^{i\alpha\phi} : \ \equiv \ e^{i\alpha\phi_{\text{creation}}}\ e^{i\alpha\phi_{\text{annihilation}}} \tag{6.194}$$

Their two-point functions are readily evaluated to give

$$\left\langle V_{\alpha,\bar{\alpha}}(z,\bar{z})V^{\dagger}_{\alpha,\bar{\alpha}}(w,\bar{w})\right\rangle = \frac{1}{(m(z-w))^{\frac{\alpha^2}{4\pi}}(m(\bar{z}-\bar{w}))^{\frac{\bar{\alpha}^2}{4\pi}}} \ . \tag{6.195}$$

Since this expression is infrared divergent, one has to renormalize the vertex operators in order to make their correlators IR finite

$$V_{\alpha,\alpha}(z,\bar{z})|_{\rm ren} \equiv m^{\frac{\alpha^2}{4\pi}} \ : e^{(i\alpha\phi(z,\bar{z}))} : \tag{6.196}$$

From (6.195) one can read off the conformal dimensions d and $\bar{d}$ of $V_{\alpha,\bar{\alpha}}$

$$d = \frac{\alpha^2}{8\pi} \qquad \bar{d} = \frac{\bar{\alpha}^2}{8\pi} \ . \tag{6.197}$$

The scaling dimension Δ and the conformal spin S are defined as $D = d + \bar{d}$ and $S = d - \bar{d}$ respectively. Below we will see how the restriction on the conformal spin to be integer or half-integer restricts the possible values of α and $\bar{\alpha}$.

Multipoint correlators are also easily evaluated and the general result is (for simplicity we take $\alpha_i = \bar{\alpha}_i$)

$$\left.\left\langle\prod_{i=1}^{N} V_{\alpha_i,\alpha_i}(z_i,\bar{z}_i)\right\rangle\right|_{ren} = \prod_{i<j}|z_i - z_j|^{\frac{\alpha_i\alpha_j}{2\pi}} \ , \quad \text{if} \quad \sum_{i=1}^{N}\alpha_i = 0 \ , \tag{6.198}$$

and zero otherwise.

The neutrality condition $\sum_{i=1}^{N}\alpha_i = 0$ is necessary for the cancellation of the renormalization constants. Otherwise the result vanishes in the zero mass limit.

Compactified Free Boson

So far we have not imposed any condition on the bosonic variable ϕ. However, in many applications in condensed matter systems, like in the XXZ chain (see discussion below (6.81)), the bosonic variable is constrained to live on a circle of radius R (usually called "compactification radius"), *i.e.* ϕ and $\phi + 2\pi R$ are identified at each space-time point. This condition restricts the allowed values for the charges α to integer multiples of $1/R$ in order for the operators to be well defined. If one further imposes that the conformal spins have to be integers (to ensure single-valuedness of correlators) then the dual field $\tilde{\phi}$ is compactified with $\tilde{R} = \frac{1}{2\pi g R}$ and the allowed charges are restricted to the set

$$\{(\alpha,\bar{\alpha})\} = \{(n/R + 2\pi g m R, n/R - 2\pi g m R), \quad n,m \ \epsilon \ \mathbb{Z}\} \tag{6.199}$$

which correpond to fields with conformal dimensions

$$h_{n,m} = 2\pi g \left(\frac{n}{4\pi g R} + \frac{1}{2} m R \right)^2 \quad , \quad \bar{h}_{n,m} = 2\pi g \left(\frac{n}{4\pi g R} - \frac{1}{2} m R \right)^2 \quad . \tag{6.200}$$

Notice that the theory is dual under the transformation $R \leftrightarrow \frac{1}{2\pi g R}$, which amounts to the interchange of the so called electric and magnetic charges (respectively n and m in (6.200)).

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