Recent results on long-range interacting systems

Self-gravitating brownian gas & the HMF model

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May 2012

Plan of the presentation

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Introduction to long-range interacting systems

- The energy is generally **extensive**, but **non additive** :
  $E_{1+2} = E_1 + E_2 + O(L^d)$, instead of $O(L^{d-1})$
- **Non equivalence of thermodynamical ensembles** at equilibrium : possible negative specific heat in the microcanonical ensemble (MCE) but not in the canonical ensemble (CE)
- Dynamics depending on the thermodynamical ensemble considered : **long-lived quasi-stationary states (QSS) in the CE**

Examples of long-range interacting systems

- Self-gravitating Newtonian (MCE) or Brownian (CE) gas :
  $\mathcal{H} = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} + \frac{G m_1 m_2}{r_{12}^{d-2}}$
- 2D turbulence : vortex interaction $\mathcal{H} = \gamma_1 \gamma_2 \ln(r_{12})$
- Mean-field models where all particles interact with each other : the HMF model

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Self-gravitating gas : equilibrium

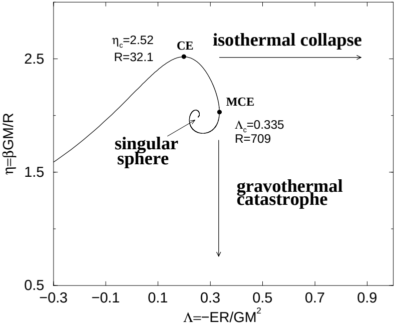
We introduce a continuous mass density $\rho(\mathbf{r})$ in a sphere of radius R, and define

- Total mass : $M = \int \rho(\mathbf{r})\, d^d r$
- Energy : $E = \frac{d}{2} MT + \frac{1}{2} \int \rho(\mathbf{r})\Phi(\mathbf{r})\, d^d r$

At equilibrium, the density is given by the **Boltzmann distribution**, where the gravitational potential is **self-consistently** obtained from **Poisson's equation** :

$$\Delta\Phi(\mathbf{r}) = G S_d \rho(\mathbf{r}),$$
$$\rho(\mathbf{r}) = Z^{-1} \exp[-\beta\Phi(\mathbf{r})].$$

+ boundary conditions (zero mass flux on the enclosing sphere)



Self-gravitating gas : dynamics in the CE

Instead of treating the dynamics of the actual Newtonian gas of particles, we assume the **existence of a large friction ξ and associated random force** (inert gas, or of effective dynamical origin...).

$$\frac{d^2\mathbf{x}_i}{dt^2} = -\xi\frac{d\mathbf{x}_i}{dt} - \nabla\Phi + \sqrt{2D\xi}\,\eta_i$$

For $\xi \to +\infty$, the problem is reduced to the dynamics of self-gravitating Brownian particles. We consider the general case $D = T\rho^{1/n}$.

The Schmoluchowski-Poisson equation (SPE) reads :

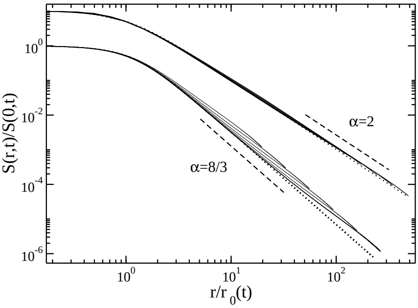
$$\frac{\partial\rho}{\partial t} = \nabla\left[\frac{1}{\xi}(D\nabla\rho + \rho\nabla\Phi)\right], \qquad \Delta\Phi(\mathbf{r}) = GS_d\rho(\mathbf{r}).$$

The constraints are :

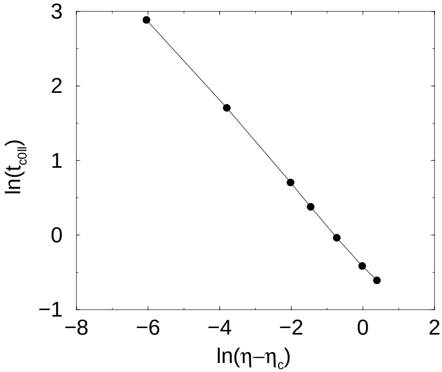
- Constant total mass M in the box of radius R
- Constant and uniform temperature T (canonical ensemble)

From now : $G = M = R = \xi = 1$

Finite-time self-similar collapse for $T < T_c$ ($D \sim T\rho^{1/n}$)



We plot $\rho(r,t)/\rho(0,t)$ as a function of $r/r_0(t)$ for different times (density range $10^2 - 10^7$) for $\alpha = 2$ and $\alpha = 8/3$ ($\alpha = \frac{2n}{n-1}$, for $n = 4$), and compare the numerics to the analytical scaling solution.



We plot t_{coll} as a function of $T_c - T$; the log-log slope is close to the theoretical result $-1/2$. (coefficient exactly known)

Dynamical phase diagram ($D = T\rho^{1/n}$; $n_* = \frac{d}{d-2}$)

Index	Temperature	Bounded domain	Unbounded domain
$n = \infty$	$T > T_c$	Metastable equilibrium state (local minimum of free energy) : box-confined isothermal sphere	• Evaporation : asymptotically free normal diffusion (gravity negligible) • Collapse : pre-collapse and post-collapse as in a bounded domain
	$T < T_c$	Self-similar collapse with $\alpha = 2$ and self-similar post-collapse leading to a Dirac peak of mass M	
$0 < n < n_*$	$T > T_c$	Equilibrium state : box-confined (incomplete) polytrope	Equilibrium state : complete polytrope (compact support)
	$T < T_c$	Equilibrium state : complete polytrope (compact support)	
$n_* < n < \infty$	$T > T_c$	Metastable equilibrium state (local minimum of free energy) : box-confined polytropic sphere	• Evaporation : asymptotically free anomalous diffusion (gravity negligible) • Collapse : pre-collapse and post-collapse as in a bounded domain
	$T < T_c$	Self-similar collapse with $\alpha = 2n/(n-1)$ and post-collapse leading to a Dirac peak of mass M	
$n = n_*$	$T > T_c$	Equilibrium state : box-confined (incomplete) polytrope	Self-similar evaporation modified by self-gravity
	$T < T_c$	Pseudo self-similar collapse leading to a Dirac peak of mass $(T/T_c)^{d/2} M$ + halo. This is followed by a post-collapse leading to a Dirac peak of mass M	Collapse
	$T = T_c$	Infinite family of steady states	Infinite family of steady states

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The HMF model

- The HMF model consists in N particles of unit mass moving on a ring and interacting via a cosine potential (Messer & Spohn 1982, Antoni & Ruffo 1995). Equivalently, it is the **mean-field version of the XY model**

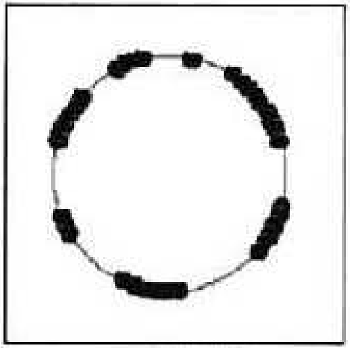
$$\frac{d\theta_i}{dt} = \frac{\partial H}{\partial v_i}, \qquad \frac{dv_i}{dt} = -\frac{\partial H}{\partial \theta_i}$$

$$H = \sum_{i=1}^{N} \frac{v_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} [1 - \cos(\theta_i - \theta_j)]$$

where θ_i and v_i denote the position (angle) and the velocity of particle i.

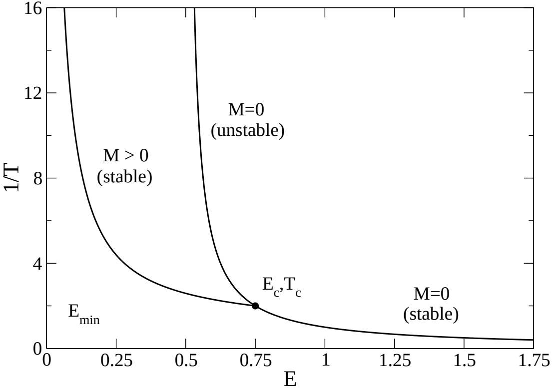
- The coupling constant $\sim 1/N$ in order to make the energy extensive
- The proper description is based on the microcanonical ensemble (MCE)
- Magnetization $M_x = \frac{1}{N}\sum_{i=1}^{N} \cos\theta_i$, $M_y = \frac{1}{N}\sum_{i=1}^{N} \sin\theta_i$

The HMF model



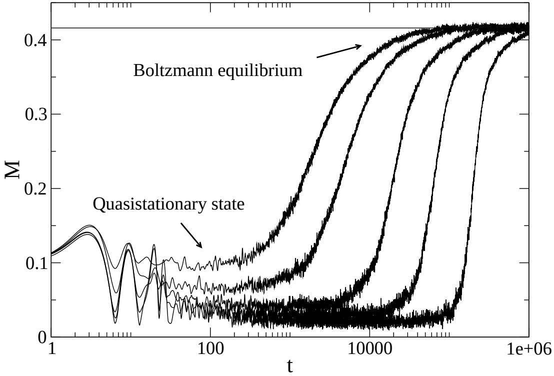
Thermodynamical equilibrium of the HMF model

- At thermodynamical equilibrium
  $f(\theta, v) = Z^{-1} \exp\left[-\beta\left(\frac{v^2}{2} + 1 - M\cos(\theta)\right)\right]$, with $M = \langle\cos(\theta)\rangle$
- For $E/N < e_c = 3/4$ (or $T < T_c = 1/2$), $M > 0$ (second order phase transition)



Two types of relaxation (collisionless vs collisional)

- Numerical simulations show that the system undergoes **two successive types of relaxation** : a violent collisionless relaxation towards a non-Boltzmannian quasistationary state (QSS) followed by a slow collisional relaxation towards the Boltzmann distribution.



Vlasov equation : life-time of the QSS

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial \theta} - \frac{\partial \Phi}{\partial \theta}\frac{\partial f}{\partial v} = O(N^{-\alpha}) \qquad \text{(Collision terms)},$$

with $\Phi(\theta, t) = \int_0^{2\pi} [1 - \cos(\theta - \theta')]\rho(\theta', t)\, d\theta'$.

The relaxation time increases with N like $t_R \sim N^{\alpha}$, depending on the homogenous/magnetized nature of the initial state and the equilibrium state :

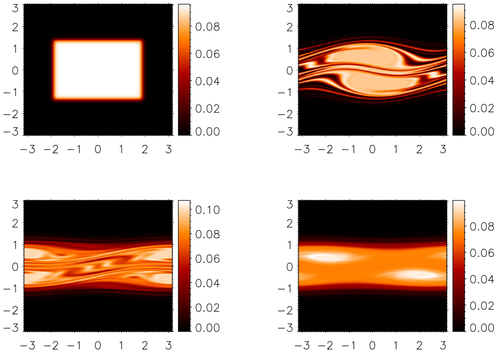
- Magnetized → Magnetized : $\alpha = 1$
- Homogenous → Homogenous : $\alpha = 2$ (???)
- Homogenous → Magnetized : $\alpha \approx 1.7$

The "final" distribution $f(\theta, v)$ in the QSS, and hence the **out of equilibrium phase diagram** (magnetized/homogenous QSS) **strongly depends on the initial distribution** $f_0(\theta, v)$.

Need to characterize the QSS and their distribution in phase space ($N \to \infty$ before $t \to \infty$)

Violent relaxation of the Vlasov equation : quasistationary state (QSS)

- Numerical simulations of the Vlasov equation from an unstable or unsteady initial condition (e.g. waterbag) show a process of phase mixing and violent relaxation towards a quasistationary state



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Initial state

We assume a initial distribution at $t = 0$, $f_0(\theta, v) = \rho_0(\theta)\sigma_0(v)$, with **high internal energy** $E \gg E_c = 3N/4$ and **non zero initial magnetization** $(M_x = M_0 > 0,\ M_y = 0)$, $M_0 = \int_{-\pi}^{\pi} \rho_0(\theta) \cos(\theta)\, d\theta$

Examples :

- $\rho_0(\theta) = \frac{1}{2\theta_0}$, for $\theta \in [-\theta_0, \theta_0]$. Then, $M_0 = \frac{\sin(\theta_0)}{\theta_0}$
- $\sigma_0(v) = \frac{\exp\left(-\frac{v^2}{2v_0^2}\right)}{\sqrt{2\pi} v_0}$ (Gaussian) ; $E_K = N\frac{v_0^2}{2}$
- $\sigma_0(v) = \frac{1}{2v_0}$, for $v \in [-v_0, v_0]$ (waterbag) ; $E_K = N\frac{v_0^2}{6}$

The total energy is $E = E_k + \frac{1-M_0^2}{2} \gg E_c$. The asymptotic distribution in the QSS takes the form (homogenous QSS) :

$$f(\theta, v) = \frac{\sigma(v)}{2\pi}$$

Idea of the calculation : importance of the smoothness of the initial velocity distribution $\sigma_0(t)$

- At high initial kinetic energy

$$\theta_i(t) \approx \theta_i(0) + v_i(0)t + \text{corrections}$$

- The magnetization is hence $M(t) = \langle \cos[\theta_i(t)] \rangle \approx M_0 \hat{\sigma}_0(t)$
  - Gaussian $\sigma_0(v)$: $M(t) = M_0 \exp(-v_0^2 t^2/2)$
  - Waterbag $\sigma_0(v)$: $M(t) = M_0 \frac{\sin(v_0 t)}{v_0 t}$
- Solve perturbatively or self-consistently the equation of motion $\ddot{\theta}_i = M \sin(\theta_i)$ and deduce the resulting velocity distribution $\sigma(v)$

- For smooth $\sigma_0(v)$ (e.g. Gaussian), the magnetization $M(t)$ decays fast and the procedure converges nicely, even for moderate $E > E_c$.
- For **singular** $\sigma_0(v)$ (e.g. waterbag), we will show that **a finite fraction $\varepsilon(E)$ of the spins synchronize with $M(t)$**, which ultimately oscillates with a finite amplitude $\sim \varepsilon(E)$

Smooth initial velocity distribution $\sigma_0(t)$

- For $M_0 = 1$ ($\theta_0 = 0$), the resulting distribution takes a particularly simple form : $\sigma(v) = \sigma_0(w(v))w'(v)$, with

$$w(v) = v\sqrt{1 + 2\int_0^{+\infty} \hat{\sigma}_0(t)\frac{\sin(vt)}{v}\, dt}$$

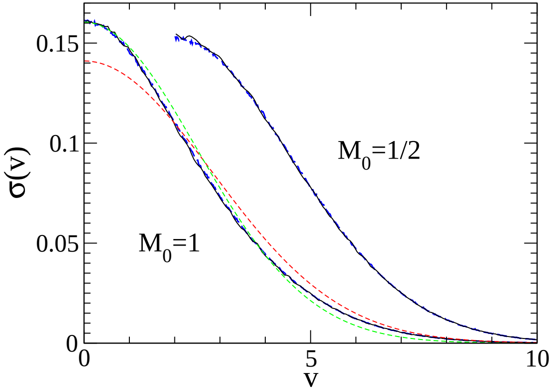
- In the Gaussian case, with $v_0 = \sqrt{2E}$ and $\sigma_0(v) = \frac{\exp\left(-\frac{v^2}{2v_0^2}\right)}{\sqrt{2\pi}v_0}$, we obtain

$$\int_0^{+\infty} \hat{\sigma}_0(t)\frac{\sin(vt)}{v}\, dt = \frac{1}{v_0\, v}\mathrm{e}^{-\frac{v^2}{2v_0^2}} \int_0^{\frac{v}{v_0}} \mathrm{e}^{\frac{u^2}{2}}\, du$$

and

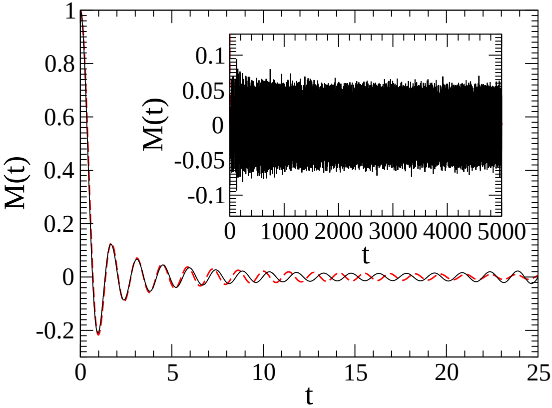
$$w(v) = v + \frac{1}{v_0}\mathrm{e}^{-\frac{v^2}{2v_0^2}} \int_0^{\frac{v}{v_0}} \mathrm{e}^{\frac{u^2}{2}}\, du + O(1/v_0^2)$$

Numerical results (Gaussian initial distribution ; $E = 4N$; $M_0 = 1/2$ and $M_0 = 1$)



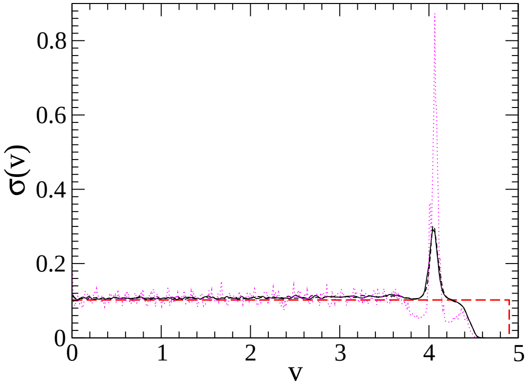
Comparison between numerical simulations of the HMF model (black ; $N = 4000000$) and the present theory (blue). The initial distribution (red) and the Gaussian exact fit at $v = 0$ (green) are also shown

Numerical results (waterbag initial distribution ; $E = 4N$; $M_0 = 1$)



Magnetization $M(t)$ (black) on a short time interval, along with a fit to $M(t) = \frac{\sin(\bar{v}t)}{\bar{v}t}$ (red). The insert shows that $M(t)$ does not ultimately decay, **oscillating with a finite amplitude $\varepsilon(E)$**.

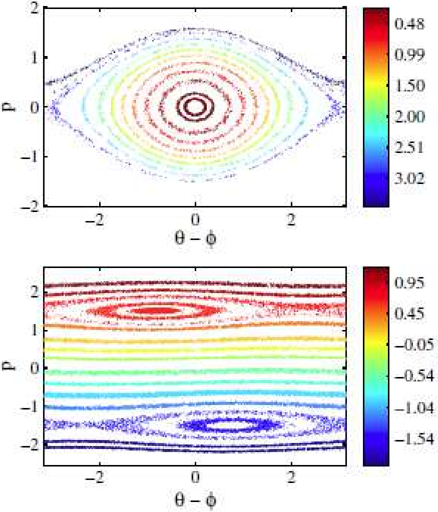
Numerical results (waterbag initial distribution ; $E = 4N$; $M_0 = 1$)



QSS velocity distribution of the HMF model (black ; $N = 1000000$ and $N = 4000000$) compared to the initial waterbag distribution (red). The distribution of the average velocities $t \in [5000 - 10000]$ is also shown (pink), revealing the delta peak at $\bar{v}$, and the velocity gap around $\bar{v}$.

Understanding of the emergence of bi-clusters

- Presence of regular tori in phase space (Bachelard *et al.* - 2008)



Understanding of the emergence of bi-clusters as a synchronization phenomenon

- The slow initial decay of $M(t)$ allows **a fraction $\varepsilon(E)$ of the particles to synchronize** with it, hence producing a magnetization $M(t) \sim \varepsilon \cos(\bar{v}t)$
- For particles with an initial velocity close to $\pm\bar{v}$, the equation of motion reads

$$\ddot{\theta} = M \sin(\theta) \approx \varepsilon \cos(\bar{v}t) \sin(\theta)$$

If $z = \theta - \bar{v}t$, then,

$$\ddot{z} \approx \varepsilon \cos(\bar{v}t) \sin(\bar{v}t + z) = \frac{\varepsilon}{2} (\sin(z) + \sin(2\bar{v}t + z)) \approx \frac{\varepsilon}{2} \sin(z)$$

- The particles hence follow a quasi-integrable motion, with Hamiltonian

$$\mathcal{H} = \frac{\dot{z}^2}{2} + \frac{\varepsilon}{2}(1 - \cos(z))$$

Understanding of the emergence of bi-clusters as a synchronization phenomenon

- The particles hence follow a quasi-integrable motion, with Hamiltonian

$$\mathcal{H} = \frac{\dot{z}^2}{2} + \frac{\varepsilon}{2}(1 - \cos(z)) = \text{constant}$$

- Assume $z(0) = 0$. If $2\mathcal{H} = \dot{z}^2(0) \leq 2\varepsilon$, z remains bounded and hence $\langle \dot{z}(0) \rangle = 0$
- This implies that the particles with initial velocity satisfying $||\dot{\theta}(0)| - \bar{v}| \leq \sqrt{2\varepsilon}$ keep the average velocity $\pm\bar{v}$.
- The number of such particles is $N_\varepsilon \approx \sigma_0(\bar{v})\sqrt{\varepsilon} \approx \frac{\sqrt{\varepsilon}}{v_0} \approx \varepsilon$

  Hence, we obtain self-consistently the density of particles synchronized to $M(t)$:

$$\varepsilon(E) \sim v_0^{-1} \sim E^{-1/2}$$

Plan of the presentation

- The non equivalence of thermodynamical ensemble is well understood as far as equilibrium properties of long-range interacting systems are concerned. However, the study of their dynamics is a very active field of research (statistical physics, astrophysics, non linear physics).
- In the canonical ensemble, the dynamics of a self-gravitating gas can be analytically studied, leading to a **complete dynamical phase diagram**.
- In the microcanonical ensemble, QSS emerge with a life time diverging with the size of the system.
- The HMF model is a simple paradigm to study the out of equilibrium dynamics of long-range interacting systems and their QSS
- We have obtained these QSS at high energy which depend strongly on the initial distribution of velocities.
- For the waterbag initial distribution, a new **partial velocity synchronization mechanism** is demonstrated, leading to the oscillation of the magnetization with amplitude $\sim E^{-1/2}$.
- Exact results can also be obtained at very low energy.

See the articles on ArXiv (*Physical Review E & Physica A*)