Anomalous diffusion of random walkers

Three examples and application to persistance

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Continuous Time Random Walk (CTRW)

$$t_n = \sum_{i=1}^{n} \tau_i , \quad x(t) = \sum_{i=1}^{\max(n:t_n \le t)} \xi_i$$

Probability distribution function of the jumps: $\phi(\xi,\tau)$

One easily gets:

$$p(x,t) = \delta(x)\Psi(t) + \int_0^t \int_{-\infty}^{+\infty} \phi(x-x',t-t')\, p(x',t')\, dt' dx'$$

$$\Psi(t) = \int_t^{+\infty} \int_{+\infty}^{+\infty} \phi(\xi,\tau)\, d\tau d\xi = \int_t^{+\infty} \psi(\tau)\, d\tau$$

Can be solved by Fourier transform in x (dual variable k) and Laplace transform in t (dual variable s)

$$\hat{p}(k,s) = \frac{1-\hat{\psi}(s)}{s} \frac{1}{1-\hat{\phi}(k,s)} ; \; \hat{\Psi}(s) = \frac{1-\hat{\psi}(s)}{s}$$

Continuous Time Random Walk (CTRW)

Independant distributions $\phi(\xi,\tau)=\lambda(\xi)\psi(\tau)$

$$\hat{p}(k,s)=\frac{1-\hat{\psi}(s)}{s}\frac{1}{1-\hat{\lambda}(k)\hat{\psi}(s)}$$

Assume

- $\psi(\tau)=\frac{1}{r}\psi_0(\tau/r)$, with $\psi_0(\tau)\underset{+\infty}{\sim}\tau^{-\beta-1}(0<\beta<1)$ and $\int_0^{+\infty}\psi_0(\tau)\,d\tau=1$

  Then, $1-\hat{\psi}(s)\underset{r\to 0}{\sim}c_1(rs)^{\beta}$

- $\lambda(\xi)=\frac{1}{h}\lambda_0(\xi/h)$, with $\lambda_0(\xi)\underset{\pm\infty}{\sim}|\xi|^{-\alpha-1}\ (0<\alpha<2)$ and $\int_{-\infty}^{+\infty}\lambda_0(\xi)\,d\xi=1$

  Then, $1-\hat{\lambda}(k)\underset{h\to 0}{\sim}c_2|hk|^{\alpha}$

Continuous Time Random Walk (CTRW)

In the limit $r, h \to 0$:

$$\hat{p}(k,s) = \frac{1-\hat{\psi}(s)}{s} \frac{1}{1-\hat{\lambda}(k)\hat{\psi}(s)} \sim \frac{c_1 r^{\beta} s^{\beta-1}}{c_1 (rs)^{\beta} + c_2 |hk|^{\alpha}}$$

To define the continuous limit, we set $c_1 r^{\beta} = c_2 h^{\alpha}$

Hence, up to a multiplicative constant

$$\hat{p}(k,s) \to \frac{s^{\beta-1}}{s^{\beta} + |k|^{\alpha}}$$

CTRW scaling distribution

$$\hat{p}(k,s) = \frac{s^{\beta-1}}{s^{\beta} + |k|^{\alpha}}$$

The inverse Laplace transforms is $E_{\beta}\left(-|k|^{\alpha} t^{\beta}\right)$,

where $E_{\beta}(z) := \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\beta n + 1)}$ is the Mittag-Leffler function

satisfying $E_{\beta}(-z) \underset{z \to +\infty}{\sim} \frac{\sin(\beta\pi)}{\pi} \frac{\Gamma(\beta)}{z}$ (for $0 < \beta < 1$)

Finally, in the continuous limit ($p(x,t=0) = \delta(x)$),

$$p(x,t) = \frac{1}{t^{\beta/\alpha}} W_{\alpha,\beta}\left(\frac{x}{t^{\beta/\alpha}}\right), \text{ with } W_{\alpha,\beta}(u) = \int_{-\infty}^{+\infty} E_{\beta}\left(-|k|^{\alpha}\right) e^{-iku} \frac{dk}{2\pi}$$

(one recovers the Gaussian distribution for $\alpha = 2$ and $\beta = 1$)

CTRW (diffusion-like) master equation

Define the Caputo fractional derivative for $0 < \beta < 1$

$$\frac{d^\beta}{dt^\beta} f(t) := \frac{1}{\Gamma(1-\beta)} \left[\frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\beta} d\tau - \frac{f(0^+)}{t^\beta} \right]$$

(related to the Riemann-Liouville derivative)

The Laplace transform of $\frac{d^\beta}{dt^\beta} f(t)$ is $s^\beta \hat{f}(s) - s^\beta f(0^+)$

Hence, the inverse Laplace transform of

$\hat{f}(s) := \hat{p}(k,s) = \frac{s^{\beta-1}}{s^\beta + |k|^\alpha}$ satisfies

$$\frac{d^\beta}{dt^\beta} f(t) = -|k|^\alpha \ f(t); \ f(t) = E_\beta\left(-|k|^\alpha \ t^\beta\right)$$

CTRW (diffusion-like) master equation

Define the Riesz fractional derivative for $0 < \alpha \le 2$

$$\frac{d^\alpha}{d|x|^\alpha} f(x) := \frac{\Gamma(\alpha+1)}{\pi} \sin\left(\frac{\alpha\pi}{2}\right) \int_0^{+\infty} \frac{f(x+\xi) + f(x-\xi) - 2f(x)}{\xi^{\alpha+1}} d\xi$$

The Fourier transform of $\frac{d^\alpha}{d|x|^\alpha} f(x)$ is simply $-|k|^\alpha \hat{f}(k)$

Hence, the inverse Laplace/Fourier transform of

$\hat{p}(k,s) = \frac{s^{\beta-1}}{s^\beta + |k|^\alpha}$ satisfies a diffusion-like equation

$\frac{\partial^\beta}{\partial t^\beta} p(x,t) = \frac{\partial^\alpha}{\partial |x|^\alpha} p(x,t)$, with solution $p(x,t) = \frac{1}{t^{\beta/\alpha}} W_{\alpha,\beta}\left(\frac{x}{t^{\beta/\alpha}}\right)$,

and $W_{\alpha,\beta}(u) = \int_{-\infty}^{+\infty} E_\beta\left(-|k|^\alpha\right) e^{-iku} \frac{dk}{2\pi}$

Other interesting processes leading to anomalous diffusion

Probability distribution function $\phi(\xi,\tau)=\lambda(\xi)\times\nu(\xi)e^{-\nu(\xi)\tau}$,

with $\nu(x)=|x|^{-\theta}\ (\theta>-1)$,

and $\lambda(\xi)=\dfrac{e^{-\xi^2/2}}{\sqrt{2\pi}}$ or $\lambda(\xi)\underset{\pm\infty}{\sim}|\xi|^{-\alpha-1}$ $(0<\alpha<2;$ Lévy)

In the continuum limit, one obtains the Fokker-Planck equation

$$\frac{\partial}{\partial t}p(x,t)=\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}\left[|x|^{-\theta}\ p(x,t)\right],$$

with a scaling solution in terms of $u=x/t^{1/(\alpha+\theta)}$

For instance, for $\alpha=2$, $p(x,t)=C_{\theta}\dfrac{|x|^{\theta}}{t^{\frac{1+\theta}{2+\theta}}}\exp\left[-\dfrac{2|x|^{2+\theta}}{(2+\theta)t}\right]$

Other interesting processes leading to anomalous diffusion

Processes defined by their Langevin-type equation of motion:

$dx = p(x,t)^{\frac{1-q}{2}} dw$, where $w(t)$ is a standard Brownian walker

(relevant in astrophysics and stock price models - option theory...)

The associated Fokker-Planck equation reads

$$\frac{\partial}{\partial t} p(x,t) = \frac{\partial^2}{\partial x^2}\left[p(x,t)^{2-q} \right],$$

with a scaling solution in terms of $u = x / t^{1/(3-q)}$:

$p(x,t) = Z(t)^{-1}\left[1+\beta(t)(q-1)x^2\right]^{-1/(q-1)}$ (Tsallis $q-$exponential)

with $Z(t) = ((2-q)(3-q)ct)^{1/(3-q)}$, and $\beta(t) = cZ(t)^{-2}$

Persistence

The persistance up to time t of a process $x(t)$ is

$$P(t) = \mathrm{Prob}(x(t') > 0, \forall t' \in [0,t])$$

Very hard problem for non Markovian processes!

(see the rest of the presentation)

More generally, we are interested in $p_>(x,t)$,

with $p_>(x,t=0) = \delta(x-x_0)$ (the distribution of "survivors")

$$P(t) = \int_0^{+\infty} p_>(x,t)\,dx$$

Persistence

Example (method of images):

$$\frac{\partial}{\partial t} p_>(x,t) = \frac{1}{2} D(t) \frac{\partial^2}{\partial x^2}\left[p_>(x,t)\right];\ p_>(x,0) = \delta(x - x_0),\ p_>(0,t) = 0$$

$$p_>(x,t) = \frac{1}{\sqrt{2\pi\sigma(t)}}\left[e^{-\frac{(x-x_0)^2}{2\sigma(t)}} - e^{-\frac{(x+x_0)^2}{2\sigma(t)}}\right],\ \sigma(t) = \int_0^t D(t')\,dt'$$

For $D(t) = 1$, and $t \to +\infty$, $p_>(x,t) = \frac{2x_0 x}{\sqrt{2\pi} t^{3/2}} e^{-\frac{x^2}{2t}};\ P(t) = \frac{2x_0}{\sqrt{2\pi t}}$

Can be easily generalized to the case $\beta = 1,\ 0 < \alpha < 2$

$$p_>(x,t) = -x_0 t^{-2/\alpha} W'_{\alpha,1}(x / t^{1/\alpha});\ P(t) \sim t^{-1/\alpha}$$

Persistence of the *q*-process

$$\frac{\partial}{\partial t} p_>(x,t) = \frac{\partial^2}{\partial x^2}\left[p_>(x,t)^{2-q} \right], \quad p_>(x=0,t)=0$$

Exact scaling solution in terms of $u = x / t^{\frac{1}{2(2-q)}}$

(instead of $u = x / t^{1/(3-q)}$ in the free case)

$$p(x,t) = t^{-\frac{1}{2-q}} F_q\left(x / t^{\frac{1}{2(2-q)}} \right); \; F_q(u) = c_1 u^{\frac{1}{2-q}} \left[1 + c_2 (q-1) u^{\frac{3-q}{2-q}} \right]^{-\frac{1}{q-1}}$$

To be compared to $c_1\left[1+c_2(q-1)u^2\right]^{-\frac{1}{q-1}}$ in the free case

(but same large u decay)

Finally, $P(t) \underset{t\to+\infty}{\sim} t^{-\frac{1}{2(2-q)}}$

Some references

- http://pre.aps.org/abstract/PRE/v69/i1/e011107
- http://pre.aps.org/abstract/PRE/v58/i2/p1621_1
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Persistence of a random temporal signal

Approximate methods and applications

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Persistence of a non-Markovian stationary Gaussian random walker

- A stationary Gaussian walker $x(\tau)$ $\left(\left\langle x(\tau)\right\rangle = 0\right)$ is *fully characterized* by $f(\tau) = \left\langle x(\tau+\tau')x(\tau')\right\rangle$

- The **persistence** is the probability that $x(s)$ *remains above (or below) a certain level* M, for all $s \in [0,\tau]$; $\mathcal{P}_<(\tau, M) \sim \exp\left(-\theta_-(M)\tau\right)$

- $\theta(M)$ is the **persistence "exponent"** probing the *full history* of the system

- $f(\tau) = \exp(-\lambda\,|\,\tau\,|)$ means that $x(\tau)$ is *Markovian*

Persistence of a non-Markovian scale-invariant random walker

- If $\langle y(t)y(t')\rangle = g(t/t')$, then $x(\tau) = y(\exp(\tau))$ is *stationary* in the *new variable* $\tau = \ln(t) \Rightarrow \mathcal{P}(\tau, M) \sim \exp(-\theta\tau) \sim t^{-\theta}$

- *Example:* the ordinary Brownian walker $W(t)$

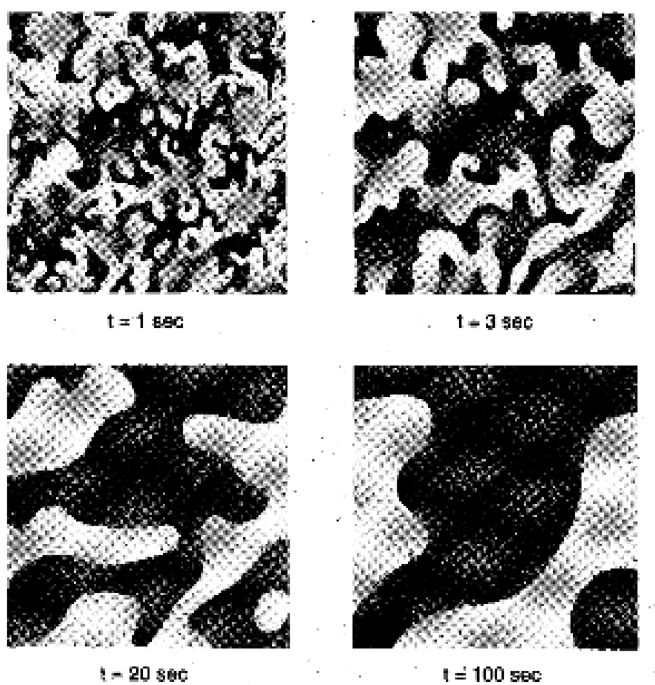
$$y(t) = W(t) / \sqrt{\langle W^2(t)\rangle}$$

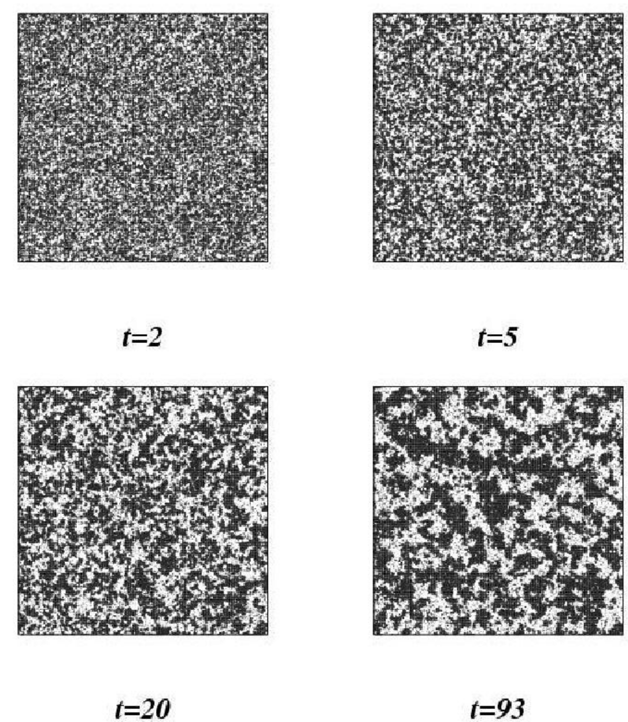
$$g(t/t') \underset{t'<t}{=} \sqrt{\frac{t'}{t}}; \; f(\tau) = \exp(-|\tau|/2); \; \theta(M=0) = \frac{1}{2}$$

- Many physical quantities of interest are (scale-invariant or stationary) *Gaussian or well approximated* by a Gaussian process

Physical applications

- **Persistence exponent of the Ising model:**
  local and global persistence; a new critical exponent for spin systems
- $\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j \,; \quad S_i = \pm 1$ on a lattice
- For $T < T_c \sim J$, the system orders at equilibrium: $\langle S \rangle \neq 0$





Physical applications: spin systems

- The total magnetization is a *true scale-invariant Gaussian process* for which the associated stationary correlator $f(\tau)$ can be computed approximately at $T < T_c$ (S. Cueille & CS) or $T = T_c$ (S. N. Majumdar *et al.*)

$$m(t) = \frac{1}{\sqrt{V}} \sum_{i \in V} S_i(t)$$

- $\mathrm{Prob}\left(m(t') > 0;\, t' \in [0,t]\right) \sim t^{-\Theta} \quad (T < T_c)$
- $\mathrm{Prob}\left(m(t') > 0;\, t' \in [0,t]\right) \sim t^{-\Theta_c} \quad (T = T_c)$

- A single spin at $T \leq T_c$ is well approximated by the *sign of a Gaussian process* (exact when $d \to \infty$)
(G. Mazenko, CS & S.N. Majumdar, S. Cueille & CS)

$$S_i(t) \approx \mathrm{sign}(y(t))$$

- $\mathrm{Prob}\left(S_i(t') > 0;\, t' \in [0,t]\right) \sim t^{-\theta} \quad (T < T_c)$
- $\mathrm{Prob}\left(S_i(t') > 0;\, t' \in [0,t]\right) \sim t^{-\theta_c} \quad (T = T_c)$

Physical applications

- Persistence exponent for the *diffusion equation* (S. N. Majumdar *et al.*, Derrida *et al.*)

$$\frac{\partial \rho}{\partial t} = \Delta_d \rho; \;\; \rho(\mathbf{x}, t=0) = random$$

$$\mathrm{Prob}\left(\rho(\mathbf{x}_0, t') > 0; \; t' \in [0, t]\right) \sim t^{-\theta_d}$$

- Persistence for any level M (CS): distribution of the *minima and maxima* of a Gaussian process

$$\mathrm{Prob}\left(\max_{t' \in [0,t]} x(t') \le M\right) = \mathrm{Prob}\left(x(t') \le M\,;\; t' \in [0, t]\right)$$

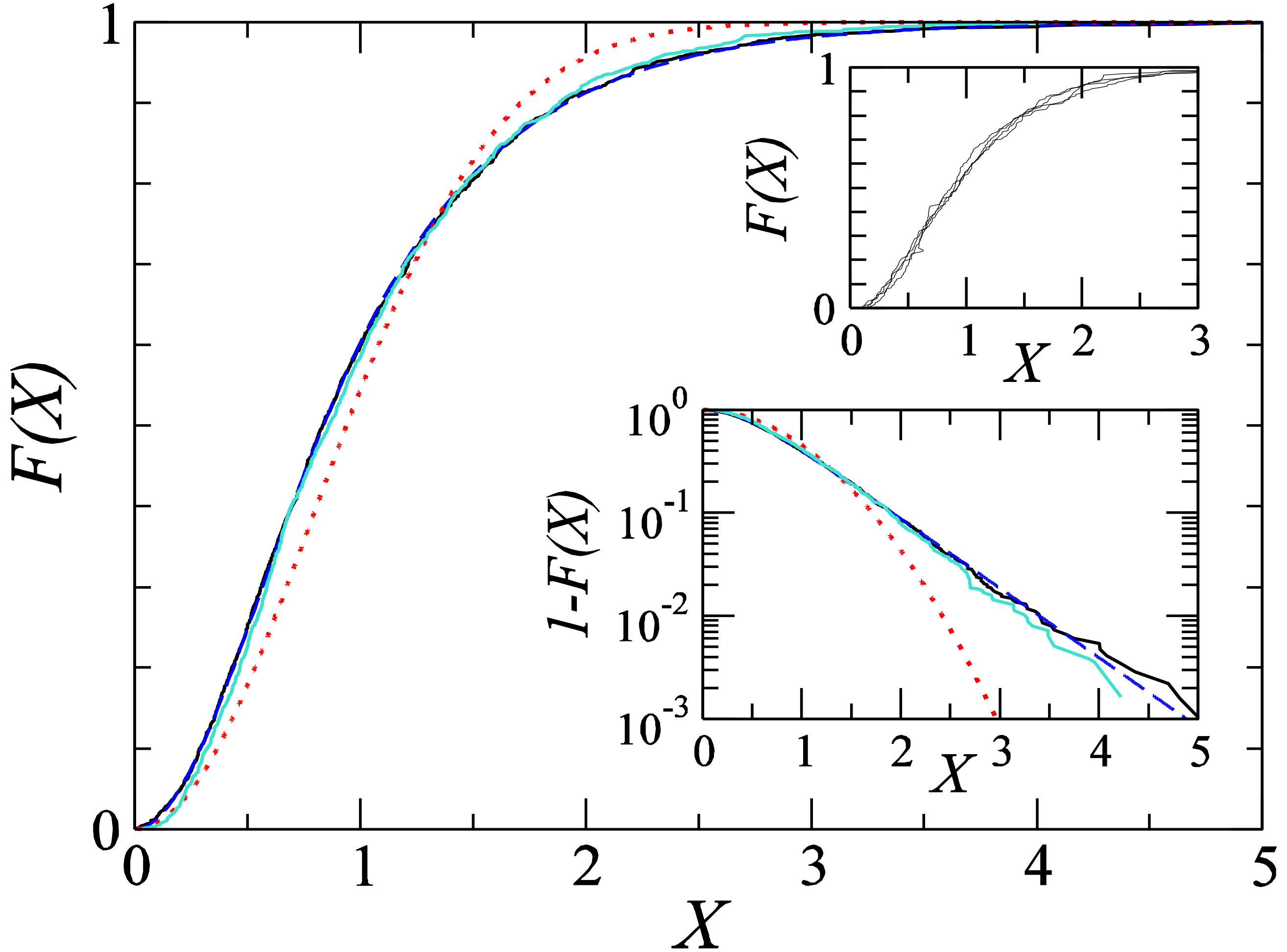
- Many other *experimental measurements* (breath figures, *Si* surfaces, soap bubbles, *Xe* gas…)

Dynamics of poker tournaments

Players survive if their fortune $x(t) > 0$ *(persistence)*

$X = x(t) / \langle x(t) \rangle =$ fortune of a player in unit of the average fortune at time *t*

$F(X) =$ fraction of players poorer than the considered player



Two approximate calculations of $\mathcal{P}(M,\tau)$

1. Perturbation theory around a Gaussian Markovian process (S. N. Majumdar & CS)

Defining the *Gaussian* weight ("action")

$$S\left(\{x(\tau)\}\right) = \frac{1}{2} \iint_{0\le t_1, t_2 \le t} x(t_1)\, f^{-1}(t_1 - t_2)\, x(t_2)\, dt_1 dt_2$$

$$\propto \frac{1}{2} \sum_{0\le i\Delta t,\, j\Delta t \le t} x_i\, f^{-1}(i\Delta t - j\Delta t)\, x_j \quad \left(x_i \equiv x(i\Delta t)\right)$$

The *persistence* is formally expressed as $(\Delta t \to 0)$

$$\mathcal{P}_<(\tau, M) = \frac{\int_{x(\tau)<M} \exp(-S\left(\{x(\tau)\}\right)\mathcal{D}x(\tau)}{\int \exp(-S\left(\{x(\tau)\}\right)\mathcal{D}x(\tau)} \qquad \left(\mathcal{D}x(\tau) \equiv \prod_{0\le i\Delta t\le t} dx_i\right)$$

The Markovian case

The *equation of motion* reads

$$\frac{dx}{d\tau}(\tau) = -\lambda x(\tau) + \sqrt{2\lambda}\,\eta(\tau) \quad \Big(dx(\tau) = -\lambda x(\tau) + \sqrt{2\lambda}\,dW(\tau)\Big)$$

And the *correlator* is

$$f_\lambda(\tau) = \langle x(\tau+\tau')x(\tau')\rangle = \exp(-\lambda\,|\,\tau\,|)\ ; \quad \hat{f}_\lambda(\omega) = \frac{2\lambda}{\omega^2+\lambda^2}$$

The "action" is really that of a *harmonic oscillator* of frequency λ (in imaginary time)

$$S_\lambda\big(\{x(\tau)\}\big) \sim \int (\omega^2+\lambda^2)\,|\,\hat{x}(\omega)\,|^2\,d\omega$$

$$\sim \int_0^\tau \left[\left(\frac{dx}{d\tau}\right)^2(\tau') + \lambda^2 x^2(\tau')\right] d\tau'$$

The Markovian case: a quantum analogy

$$\mathcal{P}_<(\tau, M) = \frac{\int_{x(\tau)<M} \exp\left(-S_\lambda\left(\{x(\tau)\}\right)\right)\mathcal{D}x(\tau)}{\int \exp\left(-S_\lambda\left(\{x(\tau)\}\right)\right)\mathcal{D}x(\tau)}$$

$$\equiv \frac{\text{partition function of QHO with a wall at } x = M}{\text{partition function of QHO at } \beta = T^{-1} = \tau}$$

$$\underset{M=0}{=} \frac{\sum_{n=1}^{+\infty} \exp\left[-\beta\left((2n-1)+\frac{1}{2}\right)\lambda\right]}{\sum_{n=0}^{+\infty} \exp\left[-\beta\left(n+\frac{1}{2}\right)\lambda\right]} \sim \exp(-\beta\lambda)$$

For a Markovian process: $\theta(M=0) = \lambda$

The nearly Markovian case for $M=0$

$$f(\tau) = \exp(-\lambda\tau) + \varphi(\tau) \quad \left(\varphi(0) = 0\right)$$

where the *perturbation* $\varphi(\tau)$ is assumed to be "small"

$$S\left(\{x(\tau)\}\right) = S_{\lambda}\left(\{x(\tau)\}\right) + \delta S_{\varphi}\left(\{x(\tau)\}\right)$$

$$\delta S_{\varphi}\left(\{x(\tau)\}\right) = -\frac{1}{2}\int \frac{\hat{\varphi}(\omega)}{\hat{f}_{\lambda}(\omega)} \, |\, \hat{x}(\omega)\,|^2 \; d\omega$$

Applying standard *quantum mechanics perturbation theory*

$$\theta = \lambda + \lim_{\tau\to+\infty} \frac{1}{\tau}\left\{ \left\langle \delta S_{\varphi}\left(\{x(\tau)\}\right)\right\rangle_{GS;\,\lambda;M} - \left\langle \delta S_{\varphi}\left(\{x(\tau)\}\right)\right\rangle_{GS;\,\lambda} \right\}$$

Results of the perturbation theory

- *Quantum perturbation theory:*

$$\theta = \lambda + \frac{2\lambda^2}{\pi}\int_0^{+\infty} \frac{\varphi(\tau)}{\left(1-\exp(-2\lambda\tau)\right)^{3/2}}\, d\tau + O\left(\phi^2\right)$$

- *Self-consistent perturbation theory:*

$$\int_0^{+\infty} \frac{f(\tau)-\exp(-\theta\tau)}{\left(1-\exp(-2\theta\tau)\right)^{3/2}}\, d\tau = 0$$

- *Resummed perturbation theory:*

$$\theta = \int_0^{+\infty} K\left(f(\tau)\right)\frac{d\tau}{\tau^2}, \quad K(u) = \frac{2}{\pi}\int_u^1 \frac{\ln^2(v)}{\left(1-v^2\right)^{3/2}}\, dv$$

Applications of the perturbation theory

- *Persistence of Ising spins at* $T < T_c$ *(exact for* $d \to \infty$*)*

$$\theta_{d=1} = \frac{3}{8} \text{ (B. Derrida)}; \;\; \theta_{Pert} = 0.35 - 0.36$$

$$\theta_{d=2} \approx 0.19 \text{ (exp.\& num.)}; \;\; \theta_{Pert} = 0.19 - 0.20$$

- *"Exact" perturbative expression of* Θ_c *at* $T = T_c$ *(S. N. Majumdar et al.; K. Oerding et al.)*

$$\Theta_c = \frac{1}{2} - \frac{1}{4}\frac{N+2}{N+8}(4-d) + a_2(N)(4-d)^2 + O\left((4-d)^3\right)$$

- *Approximate expression of* Θ $(T < T_c)$

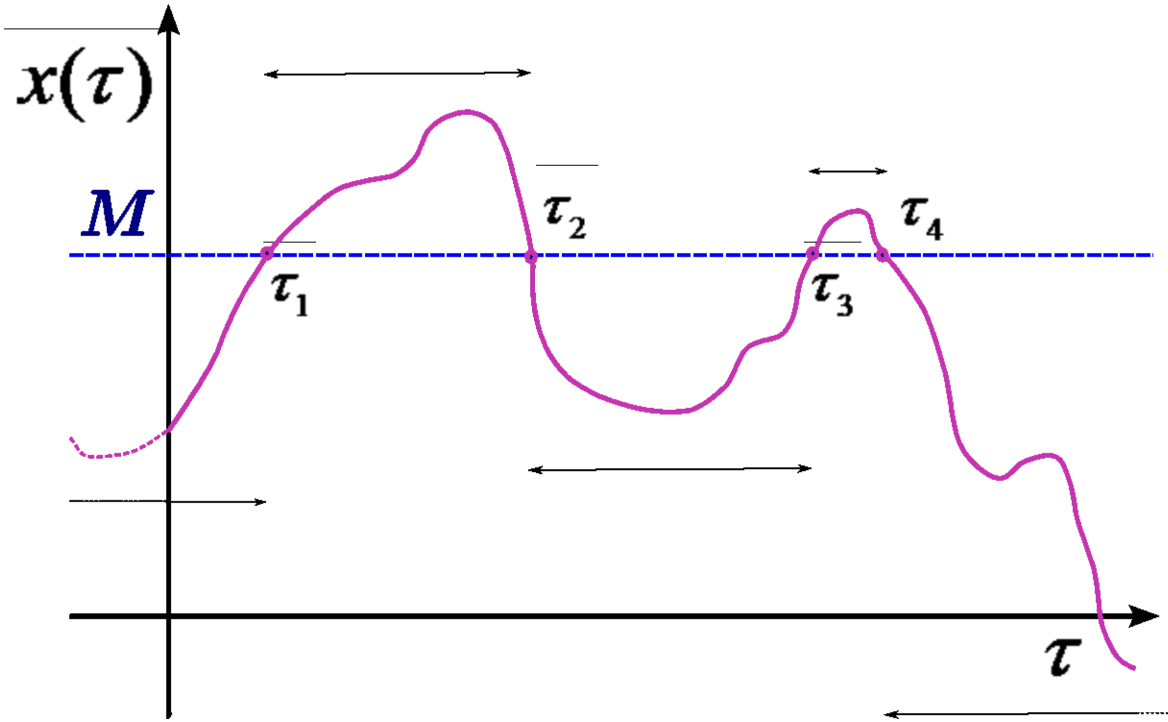
- *Exact general upper bounds for* θ*, small* $\mathcal{M}$*…*

Approximate calculation of $\mathcal{P}(M,\tau)$

2. Independent interval approximation

(S. N. Majumdar et al.; CS)

- We consider *"smooth" walkers* (continuous velocity and $f''(0)$)
- We assume **+** and **–** interval to be of *independent lengths* (drawn from two unknown distributions $P_{\pm}(\tau)$)



Independent interval approximation for any M

- Assume one knows (analytically, experimentally or numerically):
  - The *autocorrelation function*

$$A_<(\tau) = \left\langle \vartheta[M - x(\tau)]\,\vartheta[M - x(0)] \right\rangle$$

  - $N_<(\tau)$, the *average number of M-crossings* in the interval $[0,\tau]$, starting from $x(t=0) \le M$

- For any *stationary smooth Gaussian process*, both quantities can be *expressed explicitly* in terms of the correlator $f(\tau) = \left\langle x(\tau+\tau')x(\tau') \right\rangle \quad \left(f(0) = 1 \right)$

Independent interval approximation for any M

- Define $P_<(n,\tau)$ as the probability to have n *M-crossings* in the interval $\left[0,\tau\right]$, starting from $x(t=0)\le M$

- Then $N_<(\tau)=\sum_{n=0}^{+\infty} nP_<(n,\tau)$, and

$$A_<(\tau)=\left\langle \vartheta[M-x(\tau)]\,\vartheta[M-x(0)]\right\rangle = G(M)\sum_{n=0}^{+\infty} P_<(2n,\tau)$$

with

$$G(M)=1-\bar{G}(M)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{M} e^{-\frac{x^2}{2}}\,dx$$

Independent interval approximation for any M

- Using the IIA, $P_<(n,\tau)$ can be expressed in terms of $P_\pm(\tau)$

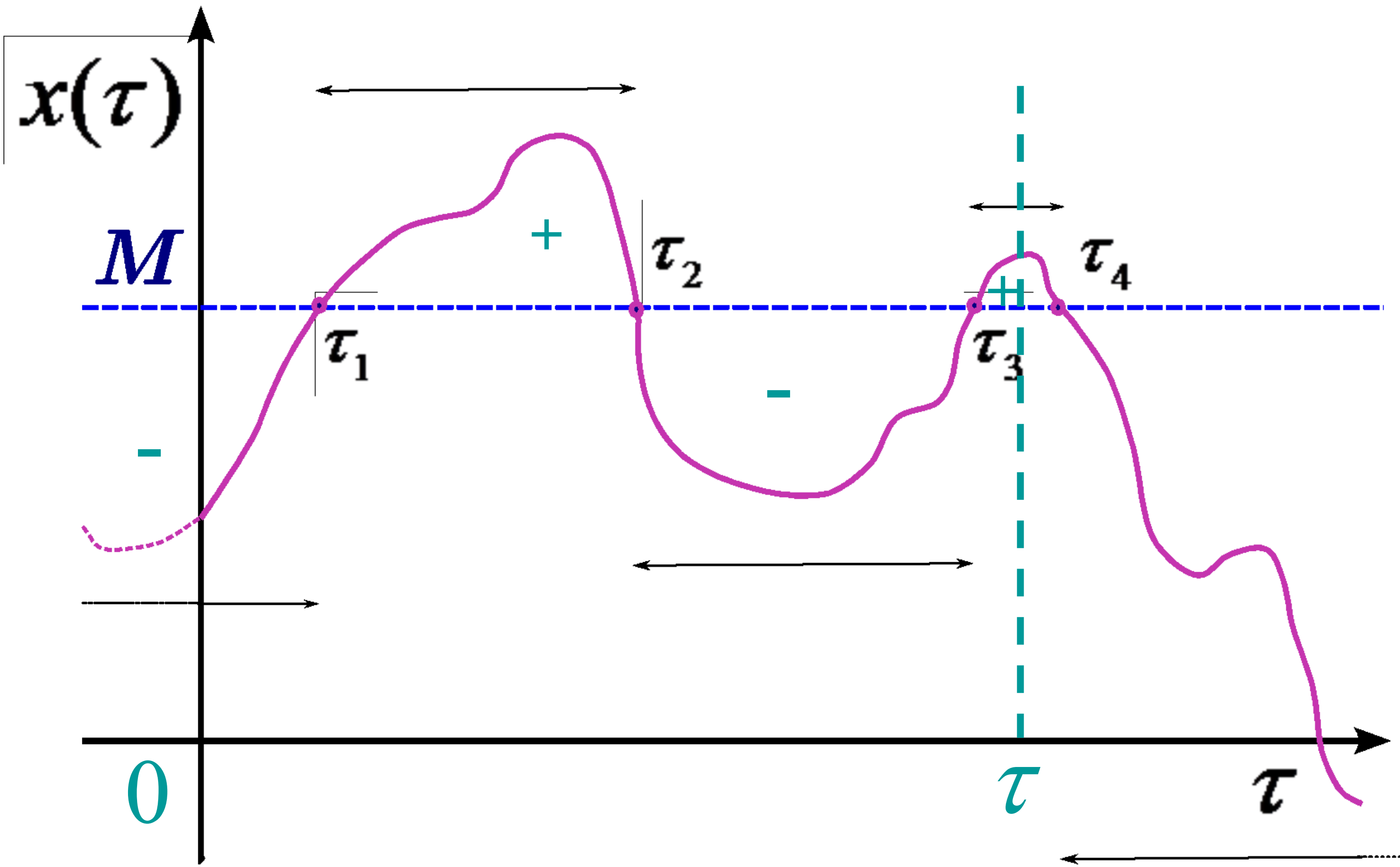
$$P_<(2n-1,\tau) = \tau_-^{-1}\int_0^\tau d\tau_1\, Q_-(\tau_1)\int_{\tau_1}^\tau d\tau_2\, P_+(\tau_2-\tau_1)\int_{\tau_2}^\tau d\tau_3\, P_-(\tau_3-\tau_2)\cdots$$

$$\int_{t_{2n-3}}^\tau d\tau_{2n-2}\, P_+(\tau_{2n-2}-\tau_{2n-3})\int_{\tau_{2n-2}}^\tau d\tau_{2n-1}\, P_-(\tau_{2n-1}-\tau_{2n-2})Q_+(\tau-\tau_{2n-1})$$

- The *average interval lengths* are $\bar{\tau} = \dfrac{\pi}{\sqrt{-f''(0)}}\,\mathrm{e}^{\frac{M^2}{2}}$ and $\tau_- = 2\bar{\tau}\,G(M);\ \ \tau_+ = 2\bar{\tau}\,\bar{G}(M)$

- The probability to find a $\pm$ *interval larger than* τ is

$$Q_\pm(\tau) = \int_\tau^{+\infty} P_\pm(\tau')\,d\tau'$$



x(τ)
M
+
τ2
τ4
+
τ1
τ3
-
-
0
τ
τ

Independent interval approximation for any M

- The persistence $\mathcal{P}_<(\tau, M)$ is *exactly* given by an identity recovered by the IIA:

$$\mathcal{P}_<(\tau, M) = P_<(n=0,\tau) = \tau_-^{-1}\int_\tau^{+\infty}(\tau'-\tau)P_-(\tau')\,d\tau'$$

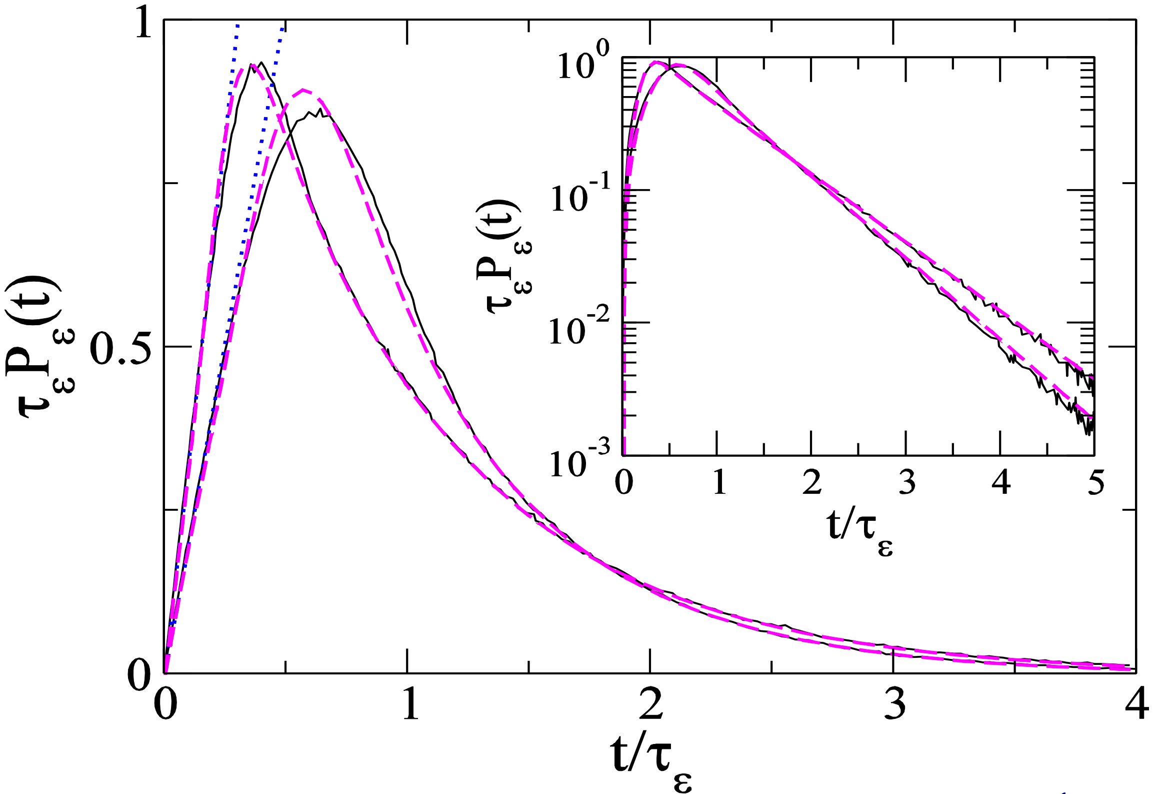
- Expressing all these equations in *Laplace space* leads to

$$\hat{N}_<(s) = \frac{(1+\hat{P}_+)(1-\hat{P}_-)}{\tau_- s^2(1-\hat{P}_+\hat{P}_-)}, \quad \hat{A}_<(s) = G(M)\left[\frac{1}{s} - \frac{1-\hat{P}_+}{1+\hat{P}_+}\hat{N}_<(s)\right]$$

$$\hat{\mathcal{P}}_<(s, M) = \frac{1}{s} - \frac{1-\hat{P}_-(s)}{\tau_- s^2}, \quad \hat{\mathcal{P}}_>(s, M) = \frac{1}{s} - \frac{1-\hat{P}_+(s)}{\tau_+ s^2}$$

Applications of the IIA

- The IIA result for $P_{\pm}(\tau)$ is exact at *small* τ up to $O\left(\tau^3\right)$ (all M) and for *large* $|M|$
- It is also very accurate for *moderate* M



$P_{\pm}(\tau)$ for $f(\tau)=\exp\left(-t^2/2\right)$ and $M=\dfrac{1}{2}$

The diffusion equation

- The associated correlator in dimension d is

$$f(\tau) = \cosh^{-d/2}(\tau/2)$$

- Comparison of the IIA to numerical simulations for $M = 0$

d	θ_{IIA}	$\theta_{\text{Num.}}$
1	0.1203…	0.1207 ± 0.0005
2	0.1862…	0.1875 ± 0.0010
3	0.2358…	0.2380 ± 0.0015

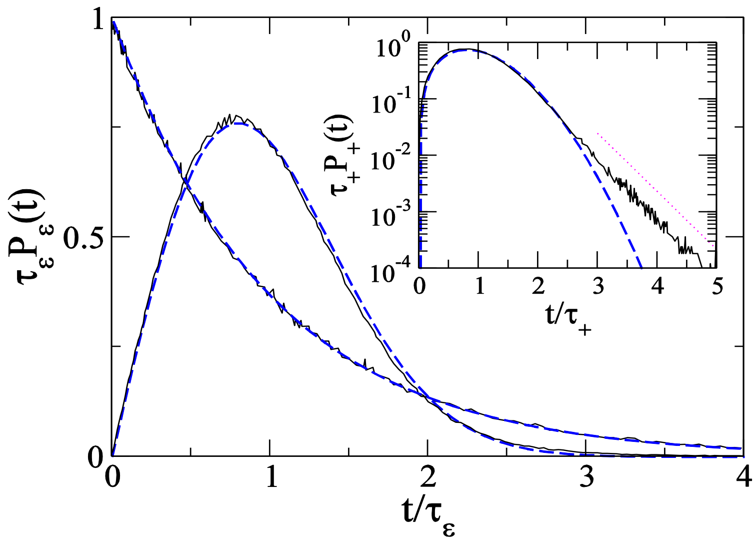
The diffusion equation

- Comparison of the IIA to numerical simulations for $d = 2$

M	$\theta_-^{\text{IIA}} \times \overline{\tau}_{M=0}$	$\theta_-^{\text{Num.}} \times \overline{\tau}_{M=0}$
-2	4.8926…	5.0(1)
-3/2	3.6677…	3.74(7)
-1	2.6475…	2.67(2)
-1/2	1.8164…	1.865(6)
0	1.1700…	1.178(2)
1/2	0.6949…	0.7008(7)
1	0.3715…	0.3743(6)
3/2	0.1734…	0.1750(4)
2	0.06813…	0.06834(6)
5/2	0.02177…	0.02180(3)
3	0.005509…	0.005510(2)

For $M \to -\infty$, $P_+(\tau) \approx \dfrac{1}{\tau_+} e^{-\frac{\tau}{\tau_+}}$

and $P_-(\tau) \approx \dfrac{\pi}{2}\dfrac{\tau}{\tau_-^2} e^{-\frac{\pi}{4}\left(\frac{\tau}{\tau_-}\right)^2}$



Other results of IIA

- The *random acceleration process* $\frac{d^2 y}{dt^2} = \eta(t)$ is associated to the *non-analytic* stationary correlator

$$f(\tau) = \frac{3}{2}\mathrm{e}^{-\frac{|\tau|}{2}} - \frac{1}{2}\mathrm{e}^{-\frac{3|\tau|}{2}}$$

For $M = 0$, $\theta = \frac{1}{4}$ (Y.G. Sinai, T.W. Burkhardt)

while $\theta_{\text{IIA}} = 0.2657...$

- *General exact bounds* for $M \geq 0$

$$\theta_{M=0} + \frac{M^2}{4\hat{f}(0)} \leq \theta_+(M) = \theta_+(-M) \leq \theta_{M=0} + \frac{\langle x(\tau)\rangle_{x>0}}{2\hat{f}(0)} M + \frac{M^2}{4\hat{f}(0)}$$

- …

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