2D Decaying Turbulence

Effective merging dynamics of two and three fluid vortices

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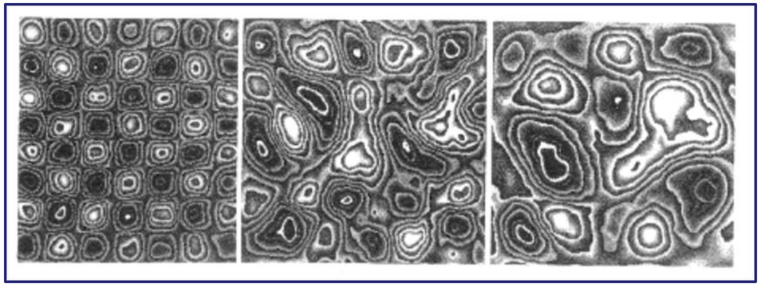
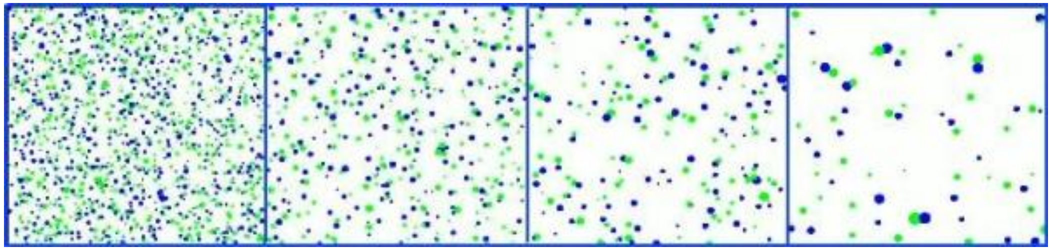
2D decaying turbulence

- Inviscid (ideally) decay of a 2D turbulent fluid
- Emergence of ***coherent vortices*** (Jupiter's red spot) followed by a ***merging dynamics*** of like-sign vortices
- Decay of the vortex density and of the vorticity moments



P. Tabeling et al.



Effective point vortex model

Euler equation NS

Phenomenology

- Effective dynamics of vortices characterized by their
  - Density $n(t) = N(t) / L^2 = R^{-2}(t)$
  - Core vorticity $\pm\omega(t)$
  - Radius $a(t)$
  - Circulation $\gamma(t) \simeq \pm\omega a^2$

- Kirchhoff's ***Hamiltonian*** equations of motion (exact for point-like vortices, *i.e.* $a \ll R$)

$$H = -\sum_{i \neq j} \gamma_i \gamma_j \ln(|\mathbf{r}_i - \mathbf{r}_j|),$$

$$\frac{dx_i}{dt} = \gamma_i^{-1} \frac{\partial H}{\partial y_i} = -\sum_{i \neq j} \gamma_j \frac{y_i - y_j}{r_{ij}^2}, \qquad \frac{dy_i}{dt} = -\gamma_i^{-1} \frac{\partial H}{\partial x_i} = \sum_{i \neq j} \gamma_j \frac{x_i - x_j}{r_{ij}^2}.$$

- ***Long range "interaction"*** (~2D electrostatics)

(G. F. Carnevale, J. C. McWilliams, Y. Pomeau, J. B. Weiss, and W. R. Young)

Merging process

- When two ***like-sign*** vortices are at a distance

  $d_{12} \sim a_1 + a_2$ (J. B. Weiss and J. C. McWilliams, R. Benzi *et al.)*

  They merge, locally ***conserving the energy***

$$E = \frac{1}{2}\int_{\text{vortex}} v^2(x,t)\, d^2x \approx \omega(t)^2 a^4(t)$$

- The ***core vorticity*** $\omega(t)$ is nearly conserved (bad mixing). Hence, the resulting vortex has a radius

$$a^4 = a_1^4 + a_2^4$$

- And the ***area fraction*** covered by the vortices decays

$$na^2 = \frac{a^2}{R^2} \xrightarrow[t\to\infty]{} 0$$

Scaling theory

- If one admits that $n(t) \sim t^{-\xi}$, then the ***conservation*** of the core vorticity ω and the total energy $E \sim n\omega^2 a^4$ implies

$$a(t) \sim t^{\xi/4} \ll R(t) \sim t^{\xi/2}, \text{ and } na^2 \sim t^{-\xi/2}$$

- In experiments, Euler equation NS, and point vortex model NS, the scaling theory is reasonably verified with

$$\xi \approx 0.6-1.0$$

- However, the asymptotic scaling regime may not be reached: $na^2 \ll 1$ (~ 0.1–0.02 in experiments and NS)

- One has $\dfrac{dn}{dt} = -\dfrac{n}{\tau_m} \Rightarrow \tau_m \sim t$

In experiments, $\tau_m \sim t^{0.57\pm0.12}$ (A. E. Hansen, D. Marteau, and P. Tabeling)

Scaling theory

- Define the ***collision time*** as $\tau \sim R / v \sim (\omega n a^2)^{-1} \sim t^{\xi/2}$
- The merging time is of the form $\tau_m \sim \dfrac{\tau}{\left(na^2\right)^{\alpha}} \sim \tau \left(\dfrac{R}{a}\right)^{2\alpha} \sim t$
- Leading to $\xi = \dfrac{2}{1+\alpha}$
- The vortex ***diffusion coefficient*** and ***velocity*** are

$$D \sim \frac{R^2}{\tau} \sim \omega a^2 \sim t^{\xi/2}, \qquad v \sim \frac{R}{\tau} \sim \frac{\omega a^2}{R} \sim \mathrm{Cste}$$

(CS & PHC)

- In the following, we express distance and time in unit of R and τ

Two-body merging process

- Consider two vortices initially at a distance of order R
- All other vortices are assumed to remain at a distance larger than R
- Expand the equation of motion of both vortices in power of

$$|\mathbf{r}_1 - \mathbf{r}_2| / R \sim a/R \ll 1$$

- The relative distance between both vortices satisfies

$$\frac{dr}{dt} = r[\cos(2\varphi)\eta_\alpha + \sin(2\varphi)\eta_\beta],$$

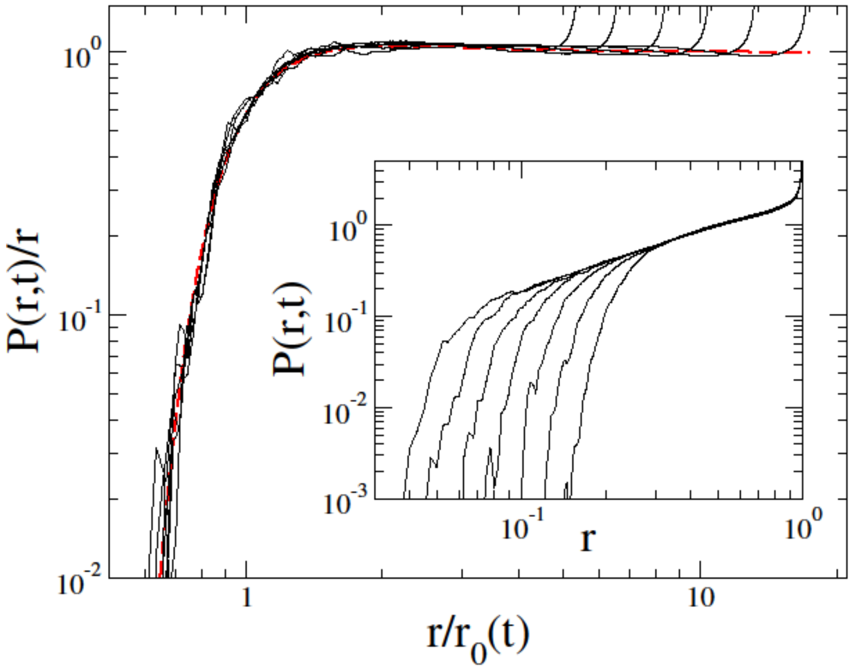
$$\frac{d\varphi}{dt} = \frac{1}{r^2} - \sin(2\varphi)\eta_\alpha + \cos(2\varphi)\eta_\beta,$$

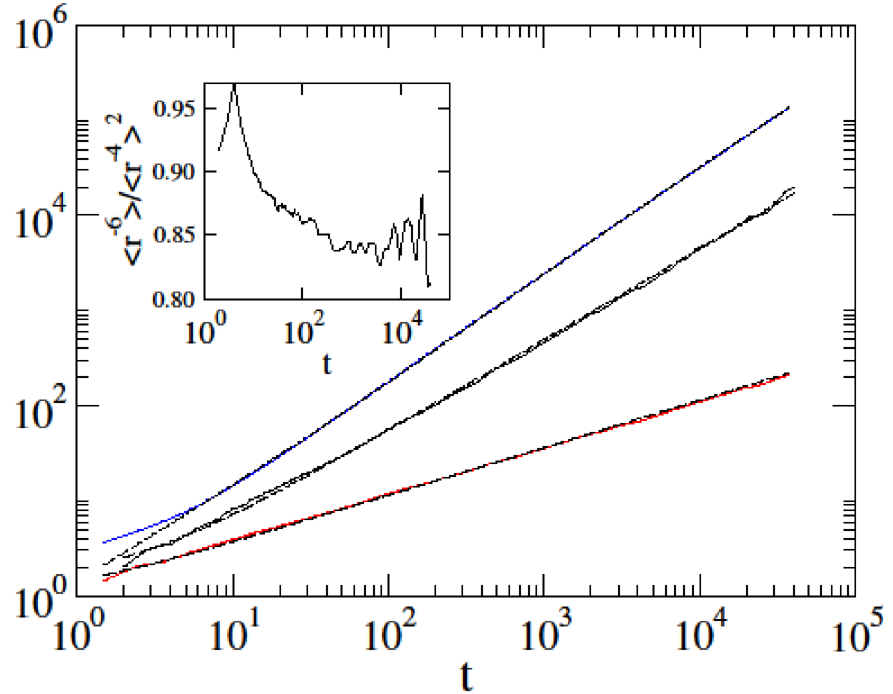
$$\eta_\alpha = \sum_{j \neq 1,2} \gamma_j \frac{2x_{0j}y_{0j}}{r_{0j}^4}, \quad \eta_\beta = \sum_{j \neq 1,2} \gamma_j \frac{y_{0j}^2 - x_{0j}^2}{r_{0j}^4}.$$

- Modelized by a ***Gaussian noise*** with $\langle \eta_\alpha^2(t) \rangle = \langle \eta_\beta^2(t) \rangle \sim \tau^{-2}$

Two-body merging process

Distribution $P(r,t) = r \times f[r / r_0(t)], \quad r_0(t) \sim t^{-1/4}, \quad f(x) \sim e^{-x^{-4}}$





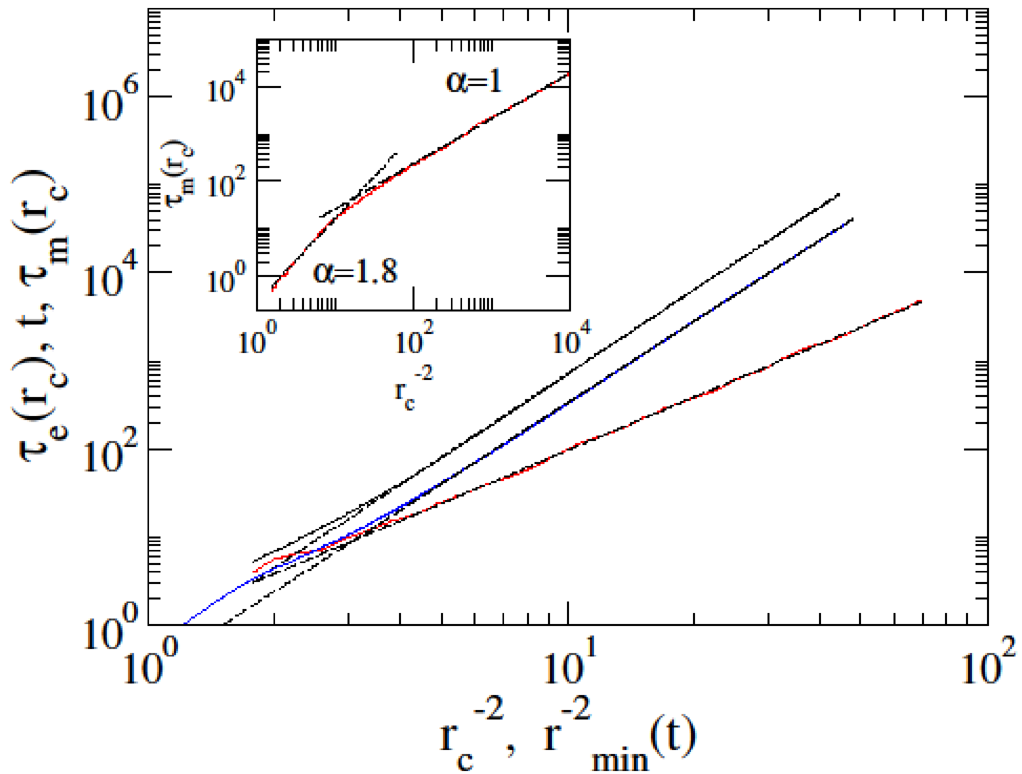
$\langle r^{-4}(t)\rangle \sim t^{1/2}, \quad \exp(\langle r^{-2}(t)\rangle), \quad \langle \varphi(t)\rangle \sim t\ln(t+t_0)$

Moments $m_z(t) = \langle r^z(t)\rangle \sim t^{-(z+2)/4}$, for $z < -2$

Merging time

- Two-body merging time
(different from the "escape time" $\tau_e(r_c) \sim r_c^{-4}$)

$$\tau_m(r_c) \sim r_c^{-6}, \quad \langle r_{\min}(t)\rangle \sim t^{-1/6} \Rightarrow \alpha = 3,\ \xi = \frac{1}{2}$$

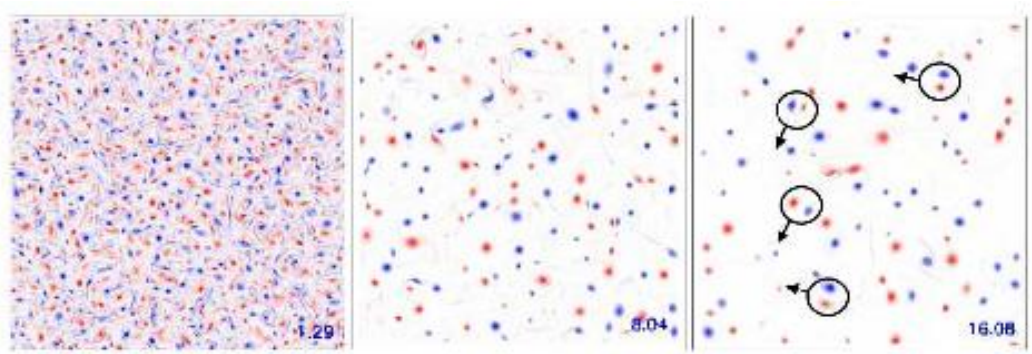


- For three-body processes: $\tau_m(r_c) \sim r_c^{-2} \Rightarrow \alpha = 1,\ \xi = 1$

Physical picture

- The fast rotation ***averages out*** the effective noise induced by the other vortices at a distance larger than $R \gg a$
- This noise (correlated over regions of size R and times of order τ) is almost the ***same for both vortices***
- The ***local energy*** $H_2 = -\gamma^2 \ln(|\mathbf{r}_1 - \mathbf{r}_2|)$ is too weakly affected by the far away vortices: quasi integrable system

- A third vortex generates a ***strong perturbation*** and the local energy conservation $H_3 \sim \gamma^2 \ln(r_{13} r_{23} / r_{12})$ allows for $r_{12} \sim a$
- At low density, mergings are dominated by the collision between ***ballistic dipoles*** and isolated vortices



$$na^2 = \frac{a^2}{R^2} \ll 1$$

Kinetic mean field theory

- A three-body merging process involves a ***dipole*** colliding with an isolated vortex

$$n_{dip} \sim n \times na^d, \quad v_{dip} \sim \gamma / a \sim \omega a$$

- Cross-section argument

$$n_{dip} a^{d-1} v_{dip} \tau_m \sim 1$$

- Scaling theory using conservation of ω and $E \sim n\omega^2 a^{d+2}$

$$\tau_m \sim \omega^{-1} \left(\frac{\omega^2}{E} \right)^{\frac{2d}{d+2}} \times n^{-\frac{4}{d+2}} \sim t \Rightarrow \xi = \frac{d+2}{4} = \frac{2}{\alpha+1}$$

$$\sim \frac{\tau}{\left(na^2\right)^{\frac{6-d}{d+2}}} \Rightarrow \alpha = \frac{6-d}{d+2}$$

Density decay exponent

- Two-dimensional turbulence $\tau_m \sim \frac{\omega}{E} \times n^{-1} \sim t \Rightarrow \xi = 1$

  Lower values found in experiments and most NS

  $(\xi \approx 0.6 - 1.0)$, ***but...*** $\tau_m \sim t^{0.57 \pm 0.12}$, and $\frac{\omega}{E} \times n^{-1} \sim t^{0.55 \pm 0.14}$

  $\xi = 1$ is found in RG simulations with moderate number of vortices but reaching $na^2 \sim 10^{-4}$ (CS & PHC)

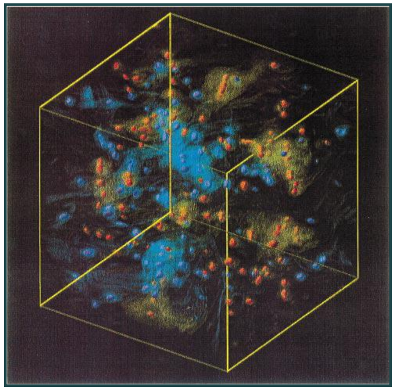
- Three-dimensional decaying geostrophic turbulence

  $\tau_m \sim \frac{\omega^{7/5}}{E^{6/5}} \times n^{-4/5} \sim t \Rightarrow \xi = 5/4$

  Compared to NS: $\xi = 1.25 \pm 0.10$

  (faster decay ; better validity of MF in high d)

  (J. C. McWilliams, J.B. Weiss, and I. Yavneh)



Conclusion

- Two-body mergings are ***inefficient at low density*** (the effective noise due to the other vortices is averaged out)
- The relevant ***three-body*** merging process involves ***ballistic dipoles*** colliding with isolated diffusing vortices
- In the scaling regime, we predict $\xi = 1$ in $d = 2$
  This result is found in RG simulations with moderate number of vortices but reaching $na^2 \sim 10^{-4}$
- Experiments and other NS methods have not yet reached low vortex densities. The ***criterion*** $\tau_m \sim t$ must be checked
- Possibility to use ***astrophysics-inspired*** NS methods
- For geostrophic turbulence in $d = 3$, we predict $\xi = 5/4$
- CS, PHC, and JS, to appear in *Phys. Rev. E*
  (preprint on ArXiv)