Autocorrelation exponent of conserved spin systems in the scaling regime following a quench

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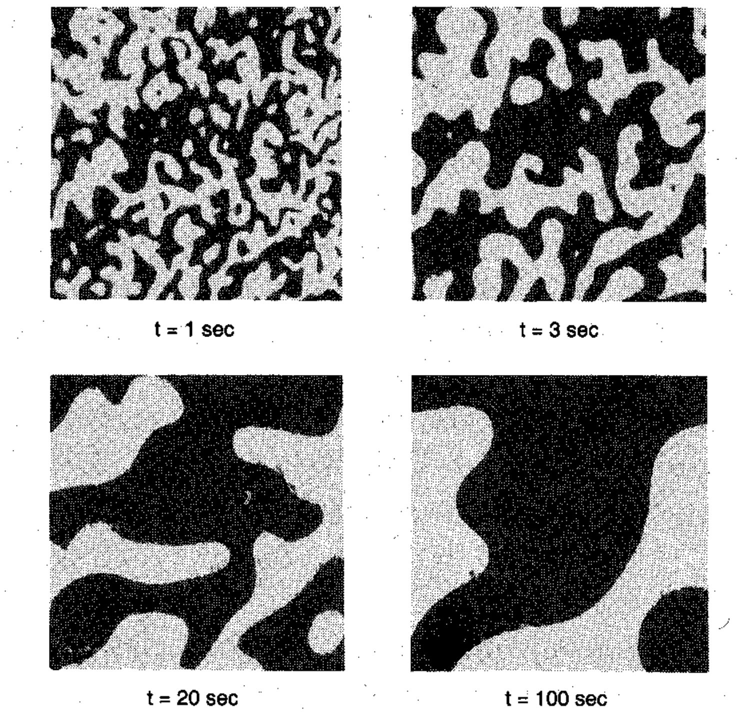
I. Introduction

General background on coarsening spin systems

- We consider a **ferromagnetic spin system** $\mathcal{H} = -\sum_{\langle i,j \rangle} \mathbf{s}(\mathbf{x}_i)\mathbf{s}(\mathbf{x}_j)$
- At $t = 0$, the system is at equilibrium at $T > T_c$
- The system is quenched at $T < T_c$ or exactly at $T = T_c$ (critical quench)
- The order parameter $\mathbf{s}(\mathbf{x}, t)$ (possibly a vector) can be
  - **non conserved** (real spin systems): ... $\uparrow$ ... $\longrightarrow$ ... $\downarrow$ ...
  - **locally conserved** (binary phase separation): ... $\uparrow\downarrow$ ... $\longrightarrow$ ... $\downarrow\uparrow$ ...
  - **globally conserved** ($\sim$ Potts model): ... $\uparrow$ ... $\downarrow$ ... $\longrightarrow$ ... $\downarrow$ ... $\uparrow$ ...
- A **coarsening length** $L(t) \sim t^{1/z}$ arises and correlation functions obey **dynamical scaling** (z is called the dynamical exponent)

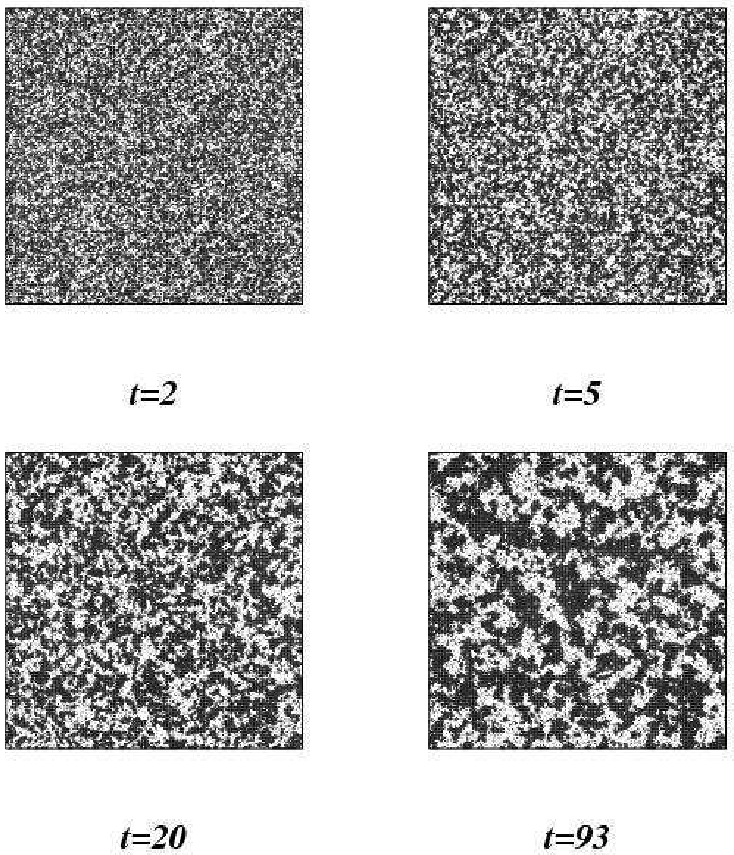
Non conserved spin systems after a quench at $T < T_c$ or $T = 0$



Right and left nematic helices play the role of up and down Ising spins in this experiment by Mason et al.

- $L(t) \sim t^{1/2}$ $(z = 2)$
- $\langle s(\mathbf{x}, t) s(\mathbf{0}, t) \rangle = f[x/L(t)]$
- $\langle s(\mathbf{x}, t) s(\mathbf{0}, t') \rangle = g[x/L(t), L(t)/L(t')]$
- **Autocorrelation function:** $\langle s(\mathbf{0}, t) s(\mathbf{0}, t') \rangle \sim [L(t)/L(t')]^{-\lambda}$
- $\lambda_{d=1} = 1$, $\lambda_{d=2} \approx 1.25$, $\lambda_d \sim d/2$ (Bray, Huse, Mazenko,...)
- Local persistence exponent: $\mathcal{P}[s(\mathbf{0}, t) s(\mathbf{0}, 0) > 0] \sim t^{-\theta}$ (Bray, Derrida, Majumdar, CS...)
- Global persistence exponent: $\mathcal{P}[M(t) M(0) > 0)] \sim t^{-\theta_g}$ (Cueille, Derrida, CS...)

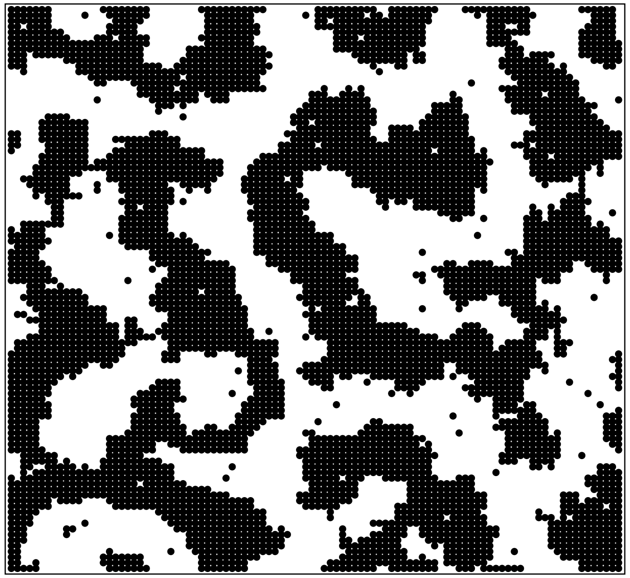
Non conserved spin systems after a quench at $T = T_c$



Coarsening of the $d = 2$ critical Ising model (Godrèche et al.)

- $L(t) \sim t^{1/z}$
  $z = 2.17$ in $d = 2$; $z = 2$ in $d \geq 4$
- $\langle s(\mathbf{x}, t) s(\mathbf{0}, t)\rangle = L(t)^{-(d-2+\eta)} f[x/L(t)]$
- **Autocorrelation function:**
  $L(t')^{d-2+\eta}\langle s(\mathbf{0}, t) s(\mathbf{0}, t')\rangle \sim \left[\frac{L(t)}{L(t')}\right]^{-\lambda_c}$
- $\lambda_{c_{d=2}} \approx 1.59$, (Huse,...)
- Global persistence critical exponent:
  $\mathcal{P}[M(t)M(0) > 0)] \sim t^{-\theta_c}$
  (Bray, Cornell, Majumdar, CS)

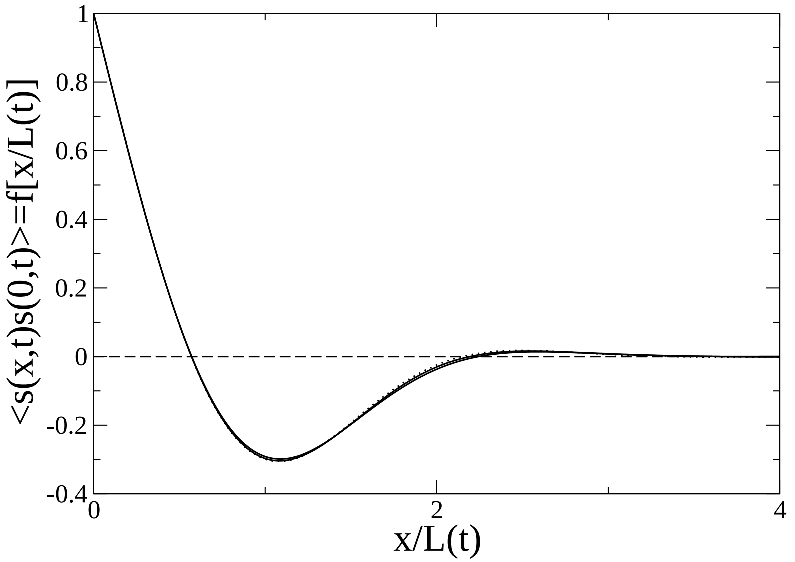
Conserved spin systems after a quench at $T < T_c$



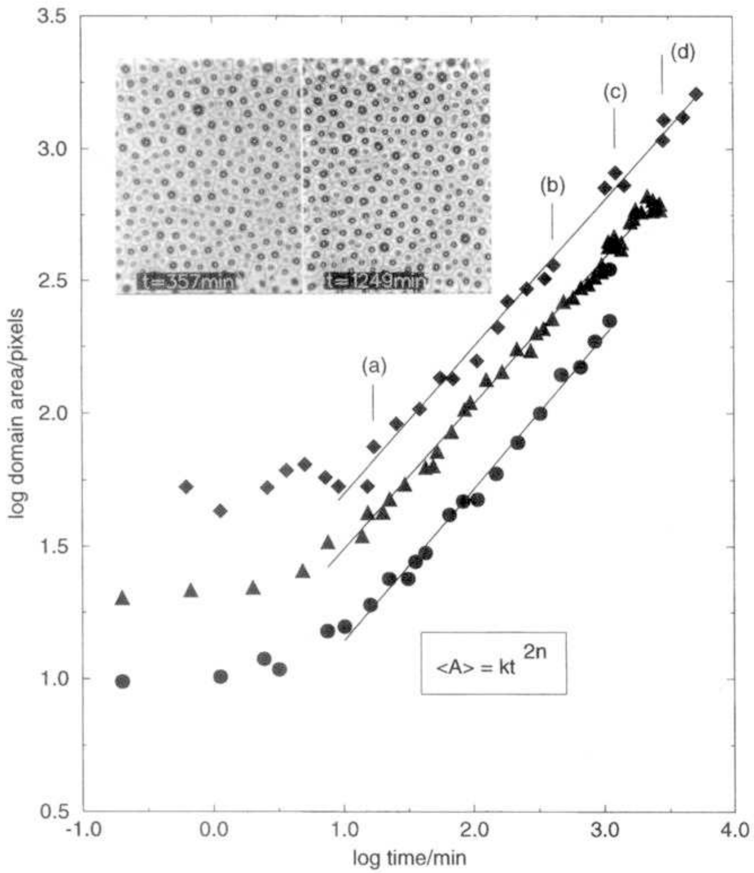
Spin exchange coarsening of the $d = 2$ Ising model at $T = 3T_c/4$

- $L(t) \sim t^{1/z}$
  $z = 3$ (scalar); $z = 4$ (vector)
- $\langle s(\mathbf{x}, t)s(\mathbf{0}, t)\rangle = f[x/L(t)]$
- $\langle s(\mathbf{x}, t)s(\mathbf{0}, 0)\rangle = L(t)^{-\lambda} g[x/L(t)]$
- **Autocorrelation function:**
  $\langle s(\mathbf{0}, t)s(\mathbf{0}, 0)\rangle \sim L(t)^{-\lambda}$
- $\lambda = d$, (Majumdar, Huse,...)
- Persistence exponents (Cueille, CS)

Conserved spin systems after a quench at $T < T_c$

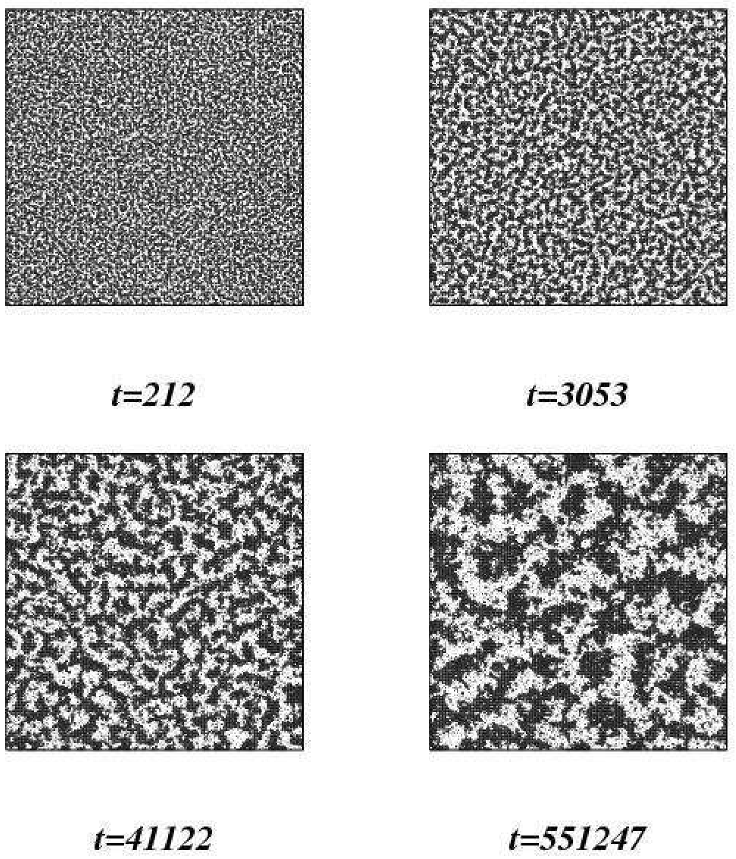


$\langle s(\mathbf{x},t)s(\mathbf{0},t)\rangle = f[x/L(t)]$ for the $d = 1$ Ising model at $T = 0$. One finds $\int f(x)\, d^d x = 0$, *i.e.* $\hat{f}(0) = 0$



Phase separation of two kinds of amphiphilic molecules on a water surface (Seul and CS): $L(t) \sim t^{1/3}$

Conserved spin systems after a quench at $T = T_c$



Coarsening of the $d = 2$ critical Ising model (Godrèche et al.)

- $L(t) \sim t^{1/z}$
  $z = 4 - \eta\,;\ \eta_{d=1} = 1,\ \eta_{d=2} = 1/4$
- $\langle s(\mathbf{x}, t) s(\mathbf{0}, t)\rangle = L(t)^{-(d-2+\eta)} f[x/L(t)]$
- **Autocorrelation function:**
  $\langle s(\mathbf{0}, t) s(\mathbf{0}, 0)\rangle \sim L(t)^{-\lambda_c}$
- $\lambda_c = d$ (Huse, Majumdar...)

Structure factor for conserved spin systems at $T \le T_c$

- $S(k,t) = \int \langle s(\mathbf{x},t) s(\mathbf{0},t)\rangle \mathrm{e}^{i\mathbf{kx}}\, d^d x$, more accessible experimentally
  - $S_s(k,t) = L(t)^{2-\eta} \hat{f}[kL(t)]$, for $T = T_c$ (scaling part)
  - $S_s(k,t) = L(t)^{d} \hat{f}[kL(t)]$, for $T < T_c$ (scaling part)

  But $S(0,t) = \left[\sum_i s_i(t)\right]^2 /N = M_0^2/N = s_0^2$, which implies $\hat{f}(0) = 0$
  Hence, $S(k,t) = S_s(k,t) + s_0^2 S_0(k,t)$

- $C(k,t) = \int \langle s(\mathbf{x},t) s(\mathbf{0},0)\rangle \mathrm{e}^{i\mathbf{kx}}\, d^d x = s_0^2 \hat{g}[kL(t)]$, with $\hat{g}(0) = 1$
  Hence, $A(t,0) = \langle s(\mathbf{0},t) s(\mathbf{0},0)\rangle \sim L(t)^{-d}$, and $\lambda = \lambda_c = d$
  (Huse, Majumdar,...)

- We shall see that $\hat{f}(k) \sim k^2$ at $T = T_c$, and
  $\hat{f}(k) \sim k^4$ ($d \ge 2$), for $T < T_c$ (Fukuwara, Yeung, CS)

Autocorrelation for conserved spin systems at $T \leq T_c$

Now, for $t \gg t' \gg t_0$, in the **scaling regime**, one expects

- For $T = T_c$,

  $$\langle s(\mathbf{x},t)s(\mathbf{0},t')\rangle = L(t')^{-(d-2+\eta)} g[x/L(t), L(t)/L(t')]$$

  $$A(t,t') = \langle s(\mathbf{0},t)s(\mathbf{0},t')\rangle \sim L(t')^{-(d-2+\eta)} \left[\frac{L(t)}{L(t')}\right]^{-\lambda'_c}$$

- For $T < T_c$,

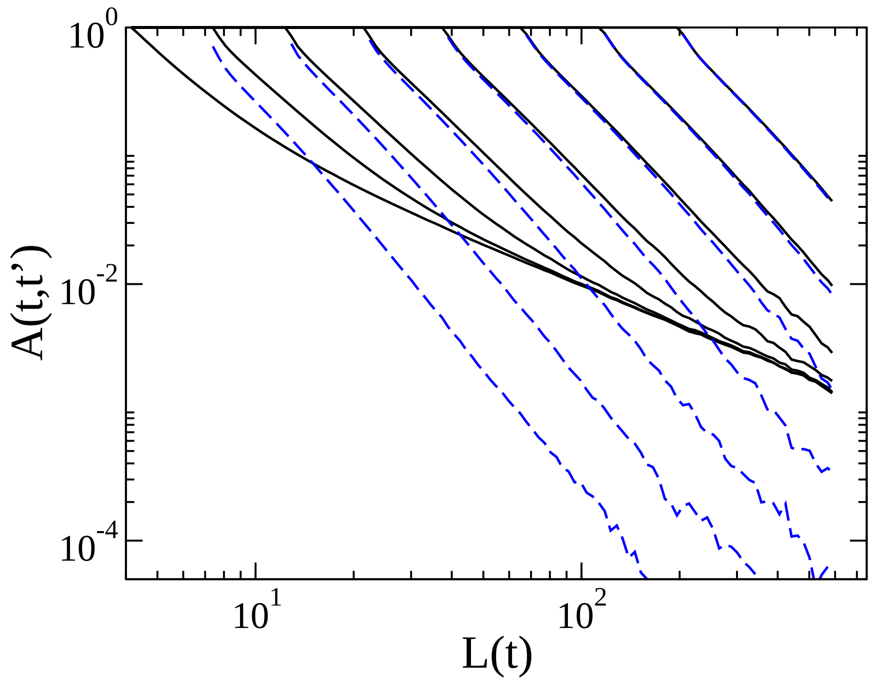
  $$\langle s(\mathbf{x},t)s(\mathbf{0},t')\rangle = g[x/L(t), L(t)/L(t')]$$

  $$A(t,t') = \langle s(\mathbf{0},t)s(\mathbf{0},t')\rangle \sim \left[\frac{L(t)}{L(t')}\right]^{-\lambda'_c}$$

In both cases, this has to be reconciled with $A(t,0) \sim L(t)^{-d}$ $(\lambda = \lambda_c = d)$

Do we have $\lambda' = \lambda$, and $\lambda'_c = \lambda_c$, like in the non conserved case ?

A new autocorrelation exponent for conserved spin systems



$A(t,t') = \langle s(\mathbf{0},t)s(\mathbf{0},t')\rangle$ as a function of $L(t)$ for several t' ($d=1$ Ising model)

$A(t,t') - A(t,0) \sim \left[\frac{L(t)}{L(t')}\right]^{-\lambda_c'}$, with $\lambda_c' \approx 3$

- For fixed t' and $t \to \infty$, $A(t,t') \sim A(t,0) \sim \frac{s_0^2}{L(t)^d}$.
- But for t' and t both in the scaling regime, $A(t,t') \sim \frac{1}{L(t')^{d-2+\eta}} \left[\frac{L(t)}{L(t')}\right]^{-\lambda_c'}$ ("$\eta = 1$" in $d=1$, $\eta = 1/4$ in $d=2$)
- In $d=1$, $\lambda_c' \approx 2.5$ (Godrèche et al.)
- In $d = 2$, $\lambda_c' \approx 3.5$ (Ising; Godrèche et al.), $\lambda_c' \approx 4$ (Cahn-Hilliard: Yeung et al.)

Experiment in $d = 1$ (Nagaya & Gilli)

FIG. 1. Time evolution of a zigzag wall. The numbers located on the left side indicate times after applying the electric field.

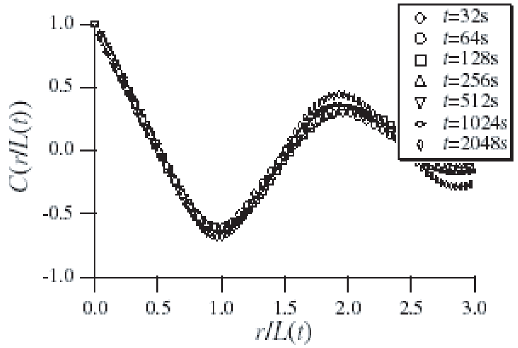


FIG. 2. Scaled spatial correlation function.

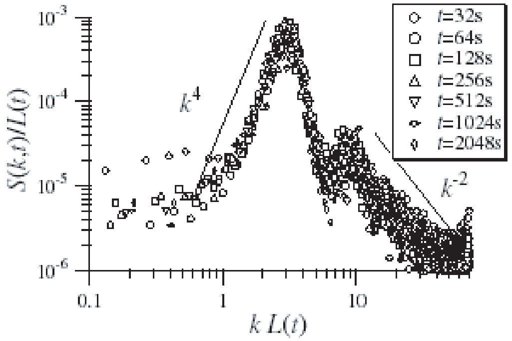


FIG. 3. Scaled power spectrum.

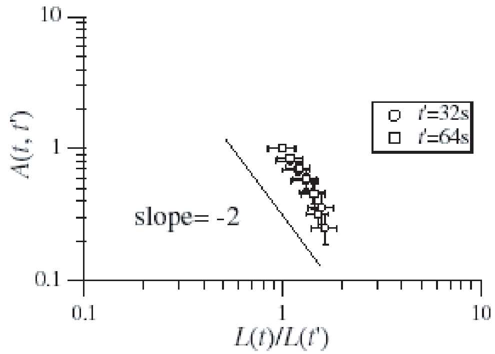


FIG. 4. Scaling behavior of two-time correlation function for the scaled characteristic time.

II. $O(n)$ Model for $n \to \infty$

The Cahn-Hilliard equation (Model B of Halperin & Hohenberg)

Consider an $O(n)$ spin $\mathbf{s}(\mathbf{x}, t) = (s_1, ..., s_n)$, associated to the energy functional

$\mathcal{H} = \frac{1}{2}(\nabla \mathbf{s})^2 + \frac{n}{4}\left(\frac{\mathbf{s}^2}{n} - k_0^2\right)^2$

$\frac{\partial \mathbf{s}}{\partial t} = -\nabla \mathbf{J}$, with $\mathbf{J} = -\nabla \mu$, and $\mu = \frac{\delta \mathcal{H}}{\delta \mathbf{s}}$

Finally, on obtains the **Model B** equation

$$\frac{\partial \mathbf{s}}{\partial t} = -\Delta \left[\Delta \mathbf{s} + k_0^2 \mathbf{s} - \frac{|\mathbf{s}|^2}{n} \mathbf{s}\right] + \eta$$

$\langle \hat{\eta}(\mathbf{k}, t) \hat{\eta}(\mathbf{k}', t') \rangle = 2Tk^2 \delta(\mathbf{k} + \mathbf{k}') \delta(t - t')$ (conserved white noise)

Compare with **Model A** which describes the Langevin dynamics of a $O(n)$ spin system ($\Leftrightarrow$ Monte-Carlo dynamics for the associated spin system)

$$\frac{\partial \mathbf{s}}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \mathbf{s}} + \eta = \Delta \mathbf{s} + k_0^2 \mathbf{s} - \frac{|\mathbf{s}|^2}{n} \mathbf{s} + \eta$$

The large n limit of the Cahn-Hilliard equation ($d > 2$)

We take the formal $n \to \infty$ limit, which implies

$$\frac{|\mathbf{s}(\mathbf{x},t)|^2}{n} = \frac{\sum_{i=1}^{n} s_i^2(\mathbf{x},t)}{n} \to \langle s^2(\mathbf{x},t)\rangle$$

where s is any component of $\mathbf{s}$. $\langle s^2(\mathbf{x},t)\rangle$ is **independent** of $\mathbf{x}$

The Cahn-Hilliard equation becomes

$$\frac{\partial s}{\partial t} = -\Delta\left[\Delta s + k_0^2 s - \langle s^2\rangle s\right] + \eta$$

which in Fourier space reads

$$\frac{\partial \hat{s}}{\partial t}(\mathbf{k},t) = -\left[k^4 - k^2(k_0^2 - \langle s^2\rangle)\right]\hat{s}(\mathbf{k},t) + \hat{\eta}(\mathbf{k},t)$$

$\langle s^2\rangle$ has to be computed self-consistently and obeys

$$\langle s^2\rangle(t) = \int^{a^{-1}} \langle s(\mathbf{k},t)s(-\mathbf{k},t)\rangle \frac{d^d k}{(2\pi)^d}$$

where a is a lattice cut-off

Structure factor (Huse, Majumdar, CS)

One finds

$S(k,t) = s_0^2 \mathrm{e}^{-2q[kL(t)]} + L(t)^{2-\eta}\hat{f}(kL(t)), \quad \text{with} \quad L(t) = t^{1/z} \quad (z = 4 - \eta = 4)$

$q(u) = u^4 - c_d u^2$ (c_d is a universal constant; $c_d = 0$, for $d > 4$), and

$$\hat{f}(u) = 2T_c u^2 \int_0^1 \mathrm{e}^{-u^4(1-v)+c_d u^2(1-\sqrt{v})}\, dv$$

In the scaling limit (*i.e.* $kL(t)$ or $x/L(t)$ of order unity)

$$S(k,t) = L(t)^{2-\eta}\hat{f}(kL(t))$$

or

$$\langle s(\mathbf{x},t)s(\mathbf{0},t)\rangle = L(t)^{-(d-2+\eta)} f[x/L(t)]$$

The term depending on initial conditions becomes negligible for large $L(t)$

Two-time correlation function (CS)

$\hat{g}(k,t,t') = \langle s(\mathbf{k},t)s(-\mathbf{k},t')\rangle$ reads

$$\hat{g}(k,t,t') = s_0^2 \mathrm{e}^{-q[kL(t')]-q[kL(t)]} + L(t')^2 \mathrm{e}^{q[kL(t')]-q[kL(t)]}\hat{f}[kL(t')]$$

For fixed t' and $t \to \infty$ (hence $L(t) \gg L(t')$), we consider the scaling limit $kL(t) \sim \mathcal{O}(1)$. Using $\hat{f}(u) \sim 2T_c u^2$, the term proportional to s_0^2 prevails (Huse, Majumdar...):

$$\hat{g}(k,t,t') \approx \hat{g}(k,t,0) = s_0^2 \mathrm{e}^{-q[kL(t)]} = s_0^2 \mathrm{e}^{-[kL(t)]^4 + c_d [kL(t)]^2}$$

Hence, in this limit $\langle s(\mathbf{x},t)s(\mathbf{0},t')\rangle = L(t)^{-d} g[x/L(t)]$, and

$$A(t,0) = \langle s(\mathbf{0},t)s(\mathbf{0},0)\rangle \sim \frac{s_0^2}{L(t)^d}$$

confirming $\lambda_c = d$

Two-time correlation function (CS)

We found $A(t,t') = A(t,0) + B(t,t') \approx A(t,0) \sim L(t)^{-d}$ for very large time. However, the second contribution can be dominant in a certain scaling time regime (note that $B(t,t) \approx k_0^2 \approx \langle s^2 \rangle$)
For $L(t) - L(t') \gg 1$, we find

$$B(t,t') = A(t,t') - A(t,0) = L(t')^{-(d-2+\eta)} C[L(t)/L(t')]$$

with

$$C(u) = u^{-d} \int^{\infty} \mathrm{e}^{-k^4(1-u^{-4})+c_d k^2(1-u^{-2})} \hat{f}(k/u) \, \frac{d^d\mathbf{k}}{(2\pi)^d}$$

$C[u] \sim a_d u^{-(d+2)}$, for $u \gg 1$, and $C[u] \sim b_d \, (u-1)^{-(d-2)/4}$, for $u - 1 \ll 1$

Hence for $L(t) \gg L(t')$, and introducing $\lambda_c' = d + 2$, we find

$$B(t,t') \sim a_d L(t')^{-(d-2)} \left[\frac{L(t)}{L(t')} \right]^{-\lambda_c'}$$

which dominates $A(t,0) \sim L(t)^{-d}$ for $L(t) \ll L(t')^{\phi}$, with $\phi = 2$

III. Phenomenological approach

The cross-over length $L_0(t) = L(t)^\phi$: a general argument

- General relation between λ'_c and ϕ:

  $L_0^{-d} \sim L^{-(d-2+\eta)}[L_0/L]^{-\lambda'_c}$, which implies

$$L_0 \sim L^\phi, \quad \text{with} \quad \phi = 1 + \frac{2-\eta}{\lambda'_c - d}$$

- Structure factor for small k:

  assuming $\hat{f}(u) \sim a_0^2 u^2$ (as found for the $O(n \to \infty)$ model; see hereafter)

  $S(k,t) = L(t)^{2-\eta}\hat{f}[kL(t)] + s_0^2 S_0(k,t) \approx s_0^2 + a_0^2 L(t)^{2-\eta}[kL(t)]^2$

- L_0 is interpreted as the **scale above which initial conditions become relevant**

  $L_0 \sim \frac{a_0}{s_0} L^\phi$, with $\phi = 2 - \eta/2 = z/2$

  which leads to the general result

$$\lambda'_c = d + 2$$

Thermal contribution to $S(k,t)$

- $L_0(t) \sim L(t)^{z/2} \sim t^{1/2}$ is simply the **diffusion scale** arising from the **thermal noise**
  Thus, the term $\sim k^2$ in $S(k,t)$ is of the form Dk^2t

- We assume that the term $\sim k^4$ in the structure factor obeys **dynamical scaling** (only even powers of k appear at small k)
  - For a quench at T_c: $S_4(k,t) = a_0^2 L(t)^{2-\eta}[kL(t)]^4$
  - For a quench below T_c: $S_4(k,t) = a_0^2 L(t)^{d}[kL(t)]^4$

A general argument for the small k behavior of $S(k,t)$

- For a quench at T_c:

  $S(k,t) \approx s_0^2 + Dk^2 t + a_0^2 L(t)^{2-\eta}[kL(t)]^4$

  $S(k,t) \approx s_0^2 + L(t)^{2-\eta}\left(D[kL(t)]^2 + a_0^2[kL(t)]^4\right)$ (using $z = 4-\eta$)

  The thermal contribution scales, which implies $\hat{f}(k) \sim k^2$

- For a quench below T_c:

  $S(k,t) \approx s_0^2 + Dk^2 t + a_0^2 L(t)^d[kL(t)]^4$

  - Scalar order parameter ($z = 3$)

    $S(k,t) \approx s_0^2 + L(t)^d\left(L(t)^{1-d}D[kL(t)]^2 + a_0^2[kL(t)]^4\right)$

    For large $L(t)$, the term $\sim k^2$ becomes irrelevant in $d > 1$

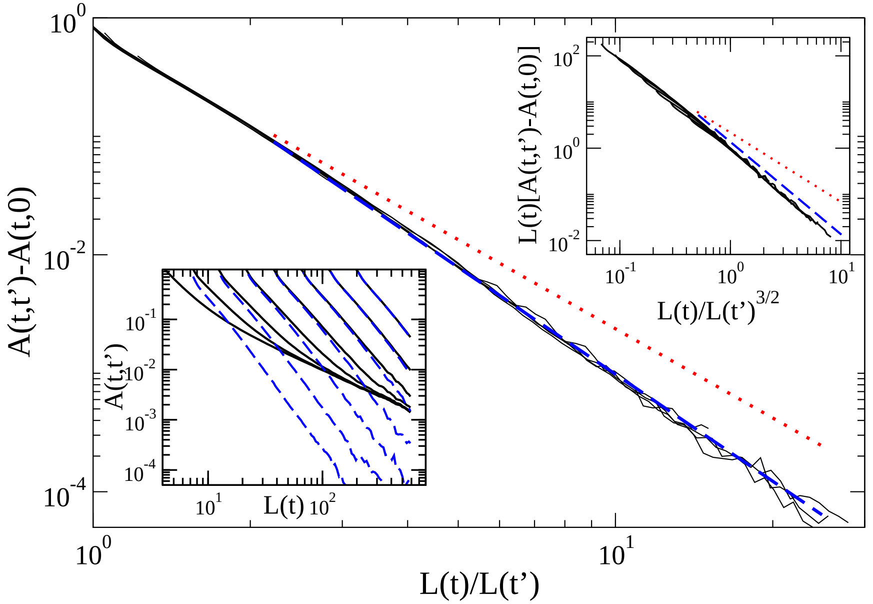
  - Vector order parameter ($z = 4$)

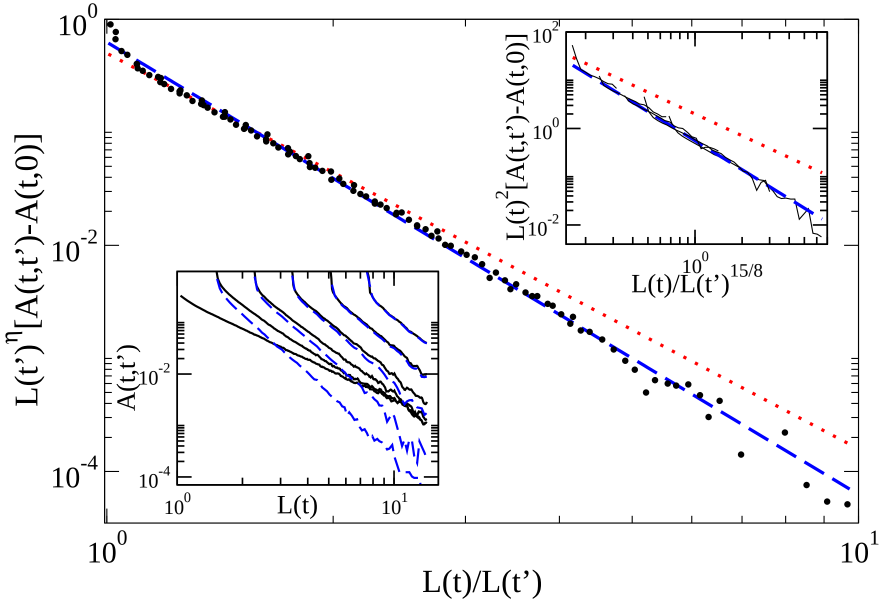
    $S(k,t) \approx s_0^2 + L(t)^d\left(L(t)^{2-d}D[kL(t)]^2 + a_0^2[kL(t)]^4\right)$

    For large $L(t)$, the term $\sim k^2$ becomes irrelevant in $d > 2$

  The thermal contribution becomes irrelevant, which implies $\hat{f}(k) \sim k^4$

Numerical simulation of the critical conserved dynamics





In $d = 1$, we plot $L(t')^{d-2+\eta}[A(t,t') - A(t,0)] = C[L(t)/L(t')]$ (formally $\eta = 1$) which decays convincingly as $[L(t)/L(t')]^{-\lambda'_c}$, with $\lambda'_c = 3$

In $d = 2$, we plot $L(t')^{d-2+\eta}[A(t,t') - A(t,0)] = C[L(t)/L(t')]$ ($\eta = 1/4$) which decays convincingly as $[L(t)/L(t')]^{-\lambda'_c}$, with $\lambda'_c = 4$

IV. Conclusion

- We have shown that

$$A(t,t') = \langle \mathbf{s}(\mathbf{0},t)\mathbf{s}(\mathbf{0},t')\rangle = s_0^2 L(t)^{-d} + L(t')^{-(d-2+\eta)} C[L(t)/L(t')]$$

  where s_0^2 measures the fluctuations in the initial state

- For $L(t') \ll L(t) \ll L(t')^{z/2} \sim t^{1/2}$, the autocorrelation is dominated by the scaling term

$$\langle \mathbf{s}(\mathbf{0},t)\mathbf{s}(\mathbf{0},t')\rangle \sim L(t')^{-(d-2+\eta)}[L(t)/L(t')]^{-\lambda_c'}$$

  with $\lambda_c' = d+2$

- The cross-over length between the two regimes is the diffusive thermal scale $L_0(t) = L(t)^{z/2} \sim Dt^{1/2}$

- The scaling part of the structure factor $\sim k^2$ at T_c, and $\sim k^4$ below T_c.

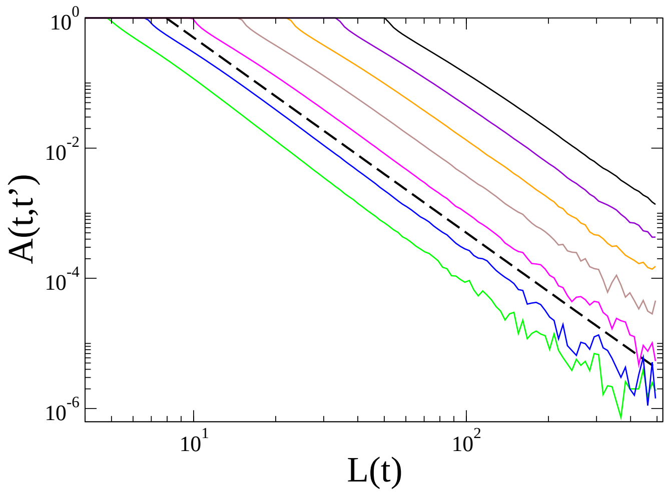
Reference: CS, cond-mat/0406333 ; Phys. Rev. Lett. **93**, 130602 (2004)

Problems for the future (J. Sopik and C. Sire, in preparation)

- For a quench below T_c and $d \geq 2$, the same argument as before leads to $\lambda' = d + 4$ and $\phi = 1 + d/4$ (scalar order parameter).

- Ising simulations at low temperature are very difficult as $L(t) \sim e^{-J/T} t^{1/3}$. Cahn-Hilliard simulations (Yeung et al.) at $T = 0$ are also difficult to conduct deep in the scaling regime (apparent λ and λ' sensitive to s_0^2, $\lambda' \sim 4 - 6$?)

  One could improve the numerical determination of λ' by subtracting $A(t, 0)$ to $A(t, t')$, or starting from initial conditions with $s_0^2 \ll 1$ as shown in the figure ($d = 1$, with $s_i(t = 0) = (-1)^i$)



- Effect of long range correlations in the initial state (as in a quench from T_c to low temperature)